Decomposition matrices for Ariki-Koike algebras and crystal isomorphisms in Fock spaces

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Résumé

Cette thèse est consacrée à l'étude des représentations modulaires des algèbres d'Ariki-Koike, et des liens avec la théorie des cristaux et des bases canoniques de Kashiwara via le théorème de catégorification d'Ariki.

Dans un premier temps, on étudie, grâce à des outils combinatoires, les matrices de décomposition de ces algèbres en généralisant les travaux de Geck et Jacon. On classifie entièrement les cas d'existence et de non-existence d'ensembles basiques, en construisant explicitement ces ensembles lorsqu'ils existent.

Ensuite, on explicite les isomorphismes de cristaux pour les représentations de Fock de l'algèbre affine quantique $\mathcal{U}_q(\widehat{\mathfrak{sl}_e})$. On construit alors un isomorphisme particulier, dit canonique, qui permet entre autres une caractérisation non-récursive de n'importe quelle composante connexe du cristal.

On souligne également les liens avec la combinatoire des mots sous-jacente à la structure cristalline des espaces de Fock, en décrivant notamment un analogue de la correspondance de Robinson-Schensted-Knuth pour le type A affine.

Mots clés : Groupe symétrique, groupe de réflexions complexes, algèbre d'Ariki-Koike, représentations modulaires, matrice de décomposition, groupes quantiques, espace de Fock, cristaux, bases canoniques, correspondance RSK.

Abstract

This thesis is devoted to the study of modular representations of Ariki-Koike algebras, and their connections with Kashiwara's crystal and canonical bases theory via Ariki's categorification theorem.

First, we study, using combinatorial tools, the decomposition matrices associated to these algebras, generalising the works of Geck and Jacon. We fully classify the cases of existence and non-existence of canonical basic sets, and we explicitly construct these sets when they exist.

Next, we make explicit the crystal isomorphisms for Fock spaces representations of the quantum affine algebra $\mathcal{U}_q(\widehat{\mathfrak{sl}_e})$. We then construct a particular isomorphism, so-called canonical, which gives, inter alia, a non-recursive description of any connected component of the crystal.

We also stress the links with the combinatorics of words underlying the crystal structure of Fock spaces, by describing notably an analogue of the Robinson-Schensted-Knuth correspondence for affine type A.

Keywords : Symmetric group, complex reflection group, Ariki-Koike algebra, modular representations, decomposition matrix, quantum groups, Fock space, crystals, canonical bases, RSK correspondence.

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Approche historique

A l'origine, une des motivations principales de l'algèbre est la résolution des équations. En particulier, des solutions explicites pour les équations polynômiales de degré 1 et 2 sont connues depuis bien longtemps, et le cas du degré 3 et 4 a été résolu à la Renaissance (par Tartaglia, Cardan et Ferrari). Au début du XIX$^{\text{ème}}$ siècle, Galois arrive à montrer, en développant les idées amorcées par Lagrange (1736-1813), l'impossibilité de résoudre par radicaux les équations polynômiales de degré au moins 5. A cette fin, il introduit le concept de groupe, en étudiant l'ensemble des permutations des racines du polynôme considéré.

Ainsi, le groupe des permutations d'un ensemble à n éléments, ou groupe symétrique $\mathfrak{S}_n$, est à l'origine de la théorie des groupes telle qu'elle s'est développée les décennies suivantes. Les groupes se définissent alors comme des structures algébriques particulières, et sont utilisés en géométrie (notamment par Klein et son "programme d'Erlangen" en 1872) ainsi qu'en théorie des nombres (par Kronecker ou encore Kummer à la fin du XIX$^{\text{ème}}$ siècle). L'étude des groupes revêt alors un intérêt indépendant, et on distingue à l'orée du XX$^{\text{ème}}$ siècle, outre les groupes de permutations, les groupes de Lie, les groupes discrets, et les groupes linéaires finis. Suite aux travaux de Coxeter (à partir des années 1920), le groupe symétrique sera vu comme un groupe de Coxeter (i.e. ayant une présentation particulière), et plus généralement comme un groupe de réflexions.

En marge de l'essor de la théorie des groupes continus pour leurs propriétés topologiques, la théorie des représentations initiée par Frobenius dès le début du XX$^{\text{ème}}$ siècle s'impose comme un outil majeur pour l'étude des groupes finis. Son objectif principal est d'obtenir des informations sur le groupe en "représentant" ses éléments par des matrices, et d'utiliser alors la théorie bien aboutie de l'algèbre linéaire. Elle participera en particulier à la classification de tous les groupes simples finis, chantier commencé dans les années 1950 et achevé en 1982. Entre-temps, des questions naturelles sont apparues : peut-on avoir une théorie complète des représentations du groupe symétrique ? Peut-on l'étendre à des groupes plus généraux ?

Enfin, notons que la théorie des représentations trouve également un intérêt dans ses applications en physique, dont l'exemple le plus frappant est le modèle de Wigner [128]. Plus précisément, à chaque état d'une particule élémentaire, libre, quantique et relativiste est associée une représentation irréductible unitaire du groupe de Poincaré.

Représentations ordinaires

D'après le théorème de Maschke, toute représentation de $\mathfrak{S}_n$ est semi-simple dès que la caractéristique de son corps de base ne divise pas l'ordre de $\mathfrak{S}_n$, à savoir $n!$ (ce qui est typiquement le cas de $\mathbb{Q}$ ou $\mathbb{C}$). Cela signifie qu'une représentation quelconque se décompose en somme directe de représentations irréductibles. Les représentations sont alors appelées *ordinaires*, et il suffit de comprendre les représentations irréductibles, ce qui était déjà le cas de Frobenius en 1901.

Dans ce contexte, il est bien connu qu'on peut construire une correspondance explicite entre les représentations irréductibles de $\mathfrak{S}_n$ et les partitions de n,

$$\begin{array}{ccc} \{\text{partitions de } n\} & \overset{1:1}{\longleftrightarrow} & \{\text{représentations irréductibles de } \mathfrak{S}_n\} \\ \lambda & \longleftrightarrow & S^\lambda, \end{array}$$

voir par exemple le livre de James et Kerber [80]. Ce paramétrage permet un traitement combinatoire systématique des questions naturelles qui apparaissent alors.

Par exemple, on peut calculer très simplement la dimension de la représentation irréductible indexée par la partition λ à l'aide de la *formule des équerres*. En représentant λ par son diagramme de Young, et en définissant l'équerre d'une boîte de λ comme le nombre de boîtes situées à sa droite ou en-dessous, la dimension de S^λ vaut alors $n!/e(\lambda)$, où $e(\lambda)$ est le produit de toutes les équerres de λ. Il existe aussi une formule explicite pour calculer les caractères, la *règle de Murnaghan-Nakayama*, voir par exemple [80, Théorème 2.4.7]. Un autre exemple est la *règle de branchement*, qui permet de calculer la décomposition de la représentation induite de S^λ (comme représentation de $\mathfrak{S}_n$) au groupe symétrique $\mathfrak{S}_{n+1}$. Celle-ci se décompose alors comme somme directe des représentations indexées par les partitions de $n+1$ obtenues à partir de λ en lui ajoutant une boîte. On peut également citer la *règle de Littlewood-Richardson*, qui permet d'expliciter la décomposition de $\mathrm{Ind}_{\mathfrak{S}_m \times \mathfrak{S}_n}^{\mathfrak{S}_{m+n}} S^\lambda \otimes S^\mu$, où S^λ (respectivement S^μ) est une représentation de $\mathfrak{S}_m$ (respectivement $\mathfrak{S}_n$). La multiplicité de la représentation de $\mathfrak{S}_{n+m}$ indexée par ν est égale au nombre de tableaux de Littlewood-Richardson de forme $\nu\backslash\lambda$ et de poids μ. On se réfèrera par exemple à [80] pour plus d'explications et d'autres résultats.

Représentations modulaires

Dès lors, une question naturelle se pose : que se passe-t-il si l'on s'intéresse à des représentations dont le corps de base est de caractéristique inférieure ou égale à n ? Dans la suite, on notera e cette caractéristique, et on parlera alors de représentations e-modulaires. Bien que le théorème de Maschke ne s'applique plus, il est possible de se restreindre à l'étude des représentations irréductibles grâce au théoreme de Jordan-Hölder, qui affirme que toutes les séries de compositions d'une représentation quelconque M sont équivalentes. Autrement dit, les représentations irréductibles apparaissant comme quotients de sous-représentations successives dans une décomposition $M \supset M_1 \supset \cdots \supset \{0\}$, ainsi que leur multiplicité, ne dépendent pas de la décomposition.

Les chapitres 5 et 6 de [80] sont une bonne référence pour la théorie des représentations e-modulaires de $\mathfrak{S}_n$. En particulier, il est désormais classique d'indexer les représentations irréductibles de $\mathfrak{S}_n$ par les partitions *e-régulières* de n, c'est-à-dire celles ne comportant pas e fois la même part. Notons alors D^μ la représentation e-modulaire irréductible indexée par la partition e-régulière μ. Dans ce cas, les problèmes classiques exposés au paragraphe précédent sont loin d'être résolus dans le cas modulaire, ne serait-ce qu'une formule pour la dimension de D^μ. On peut néanmoins, par exemple, citer les travaux de Kleshchev [92], [93], [94] sur la règle de branchement modulaire. Il y définit un processus combinatoire permettant de décrire le socle (c'est-à-dire la somme de tous les sous-modules simples) de la restriction de D^μ à $\mathfrak{S}_{n-1}$.

Les représentations irréductibles ordinaires et modulaires sont reliées par la *matrice de décomposition* de $\mathfrak{S}_n$. James and Kerber la définissent grâce à une représentation matricielle particulière : la forme naturelle de Young (voir [80, 3.4.17]), dont les coefficients sont des entiers. Il est alors possible de les réduire modulo e, ce qui définit une représentation e-modulaire. Pour une partition λ de n, on note $\overline{S^\lambda}$ la représentation ainsi obtenue à partir de S^λ. Cette représentation n'est en général plus irréductible, et l'on peut s'intéresser aux représentations modulaires irréductibles D^μ apparaissant dans sa décomposition de Jordan-Hölder. On note $d_{\lambda,\mu} = [\overline{S^\lambda} : D^\mu]$ la multiplicité correspondante et dénommée nombre de décomposition. Ces informations sont encodées dans une matrice

$$D(\mathfrak{S}_n) = (d_{\lambda,\mu})_{\substack{\lambda \text{ partition de } n \\ \mu \text{ partition } e\text{-régulière de } n}},$$

appelée matrice de décomposition. La connaissance de cette matrice permet d'obtenir en particulier les dimensions ainsi que les caractères des représentations irréductibles e-modulaires. On peut alors se demander :

Questions 1.

1. Est-il possible de trouver une forme particulière de cette matrice ?
2. Comment la calculer explicitement ?

Le théorème suivant répond en partie à la première question.

Théorème 1.

1. *$d_{\lambda,\mu} = 0$ sauf si λ et μ ont le même e-cœur.*
2. *$d_{\lambda,\mu} = 0$ sauf si μ domine λ, et $d_{\mu,\mu} = 1$ pour tout μ.*

Ici, le *e-cœur* est la partition obtenue en supprimant récursivement tous les rubans de longueur e (i.e. e boîtes adjacentes sur la frontière), et l'ordre de dominance est défini au chapitre suivant, voir le paragraphe 1.2.1. Le premier point permet d'obtenir une forme par blocs pour $D(\mathfrak{S}_n)$. Il est connu sous l'appellation "conjecture de Nakayama", bien qu'il ait été prouvé pour la première fois en 1947. Le deuxième point assure que la matrice de décomposition est unitriangulaire supérieure, à condition de bien ordonner ses lignes et ses colonnes. On dit alors que les partitions e-régulières de n forment un *ensemble basique* pour $\mathfrak{S}_n$ par rapport à l'ordre de dominance.

Groupes de Grothendieck et fonctions symétriques

Considérons l'algèbre de groupe $F\mathfrak{S}_n$, où F est un corps. Dans le contexte de représentations d'algèbres et non plus de groupes, on privilégiera la terminologie "module" et "simple" plutôt que "représentation" et "irréductible".

Soit $[F\mathfrak{S}_n]$ le groupe de Grothendieck de la catégorie des $F\mathfrak{S}_n$-modules de type fini. Notons $[M]$ l'élément de $F\mathfrak{S}_n$ associé au module M, appelé la *classe* de M. La loi de groupe est donnée par $[M] = [L]+[N]$ s'il existe une suite exacte courte $\{0\} \to L \to M \to N \to \{0\}$. Par le théorème de Jordan-Hölder, l'ensemble des classes de modules simples engendre $F\mathfrak{S}_n$ en tant que groupe abélien libre. Plus de détails concernant les groupes de Grothendieck se trouvent par exemple dans [31]. Considérons alors les groupes de Grothendieck associés à tous les groupes symétriques à la fois, et notons

$$K[F] = \bigoplus_{n\in\mathbb{Z}_{>0}} [F\mathfrak{S}_n],$$

à qui l'on donne la structure d'anneau en introduisant la multiplication définie par $[M][N] = [M \otimes N \uparrow_{\mathfrak{S}_m\otimes\mathfrak{S}_n}^{\mathfrak{S}_{m+n}}]$, pour M un $F\mathfrak{S}_m$-module et N un $F\mathfrak{S}_n$-module.

Maintenant, si $F = \mathbb{Q}$, l'anneau $[F]$ est canoniquement isomorphe à l'anneau des fonctions symétriques en une infinité de variables, noté Sym, dont une base est donnée par les fonctions de Schur s_λ, cet isomorphisme étant donné par l'application

$$[S^\lambda] \longmapsto s_\lambda.$$

Dans le cas $F = \mathbb{F}_e$, considérons l'idéal de Sym engendré par les sommes de puissances $\sum_i x_i^k$ avec e divisant k. Alors l'application

$$[\overline{S^\lambda}] \longmapsto s_\lambda \mod \mathcal{I}_e$$

induit un morphisme d'anneaux entre $[F]$ et $\mathrm{Sym}/\mathcal{I}_e$. On note $d(s_\lambda) = s_\lambda \mod \mathcal{I}_e$ l'image de $[\overline{S^\lambda}]$ par cette application, et, pour toute partition e-régulière μ, on note ϕ_μ l'image de $[D^\mu]$. On a alors

$$d(s_\lambda) = \sum_{\mu\ e\text{-régulière}} d_{\lambda,\mu}\phi_\mu,$$

où les coefficients $d_{\lambda,\mu}$ sont les nombres de décomposition définis au paragraphe précédent (voir par exemple [95, partie 2.4]). En d'autres termes, la matrice de l'application d : $\mathrm{Sym} \longrightarrow \mathrm{Sym}/\mathcal{I}_e$ relativement aux bases $\{s_\lambda\}$ et $\{\phi_\mu\}$ est la matrice de décomposition de $\mathfrak{S}_n$. On appelle alors d l'*application de décomposition*, et on la voit comme une application de $[\mathbb{Q}\mathfrak{S}_n]$ vers $[\mathbb{F}_e\mathfrak{S}_n]$.

Les questions 1 peuvent alors être reformulées en termes d'application de décomposition. D'ailleurs, la notion d'ensemble basique pour les groupes précédemment introduite est usuellement définie via l'application de décomposition : un ensemble $\mathscr{B}$ de représentations irréductibles de G en caratéristique 0 est appelé un *e-ensemble basique* si l'image des classes des éléments de $\mathscr{B}$ par l'application de décomposition d est une base de $[\mathbb{F}_eG)]$, voir [71] pour la définition originelle. Il est intéressant de noter que l'existence et la construction d'ensembles basiques pour les groupes finis est loin d'être évidente, même dans le cas des groupes classiques. On a déjà vu que le théorème , point 2, donne un ensemble basique pour $\mathfrak{S}_n$, et des résultats similaires sont connus pour les groupes de type Lie sous certaines conditions, voir [48], [44] et [45]. Les travaux récents de Brunat [17] (pour les groupes linéraires finis $\mathrm{GL}_n(q)$, $n = 2, 3, 4$), et de Brunat et Gramain [19], [18] (pour le groupe alterné $\mathfrak{A}_n$) viennent s'ajouter à cette courte liste.

Algèbre de Hecke du groupe symétrique

Pour répondre au deuxième point de la question 1, on introduit la notion d'*algèbre de Hecke*. C'est une déformation de l'algèbre du groupe symétrique $\mathbb{C}\mathfrak{S}_n$ par un paramètre v, considéré indéterminé : on la définit comme la $\mathbb{C}(v)$-algèbre associative et unitaire $\mathcal{H}_n$ engendrée par T_i, $i = 1, \dots, n-1$ et sujette aux relations

- $T_iT_j = T_jT_i$ si $|i-j| > 1$,
- $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$ pour tout $i \leq n-2$,
- $T_i^2 = v + (v-1)T_i$ pour tout $i \leq n-1$.

En spécialisant v à 1, on obtient la présentation classique de $\mathfrak{S}_n$.

Il se trouve que la théorie des représentations de $\mathcal{H}_n$ est similaire à celle de $\mathfrak{S}_n$. D'une part, quand v n'est pas spécialisé à une racine de l'unité d'ordre e, alors $\mathcal{H}_n \simeq \mathbb{C}\mathfrak{S}_n$, et on a une théorie des représentations ordinaires de $\mathcal{H}_n$: toute représentation est semi-simple, et les irréductibles sont paramétrées explicitement par les partitions de n, que l'on notera encore S^λ. D'autre part, si v est spécialisé en une racine primitive $e^{\text{ème}}$ de 1, notons-la ζ, la théorie des représentations de $\mathcal{H}_n$ se rapproche de la théorie des représentations e-modulaire de $\mathfrak{S}_n$. On se réfèrera à [32] et [33] pour plus de détails, ou encore à l'introduction de l'article de Lascoux, Leclerc et Thibon [95].

Il est possible de déterminer une forme matricielle pour les modules simples S^λ, dont les coefficients sont des polynômes de Laurent en v. On peut alors regarder la spécialistation en $v = \zeta$ de S^λ, que l'on note $\overline{S^\lambda}$. Ces modules ne sont plus simples en général. Par un théorème de Dipper et James [33], les partitions e-régulières de n indexent les modules simples de $\mathcal{H}_n$ lorsque $v = \zeta$. On les note à nouveau D^μ. On peut alors définir la matrice de décomposition de $\mathcal{H}_n$ par

$$D(\mathcal{H}_n) = (d_{\lambda,\mu})_{\substack{\lambda \text{ partition de } n \\ \mu \text{ partition } e\text{-régulière de } n}},$$

où $d_{\lambda,\mu} = [\overline{S^\lambda} : D^\mu]$ est la multiplicité D^μ dans $\overline{S^\lambda}$.

De plus, le théorème reste vrai pour $D(\mathcal{H}_n)$. Si les matrices $D(\mathfrak{S}_n)$ et $D(\mathcal{H}_n)$ ne coïncident pas en général, James montre néanmoins qu'il existe une matrice d'ajustement unitriangulaire permettant de passer de l'une à l'autre. En 1990, il conjecture même que ces deux matrices sont égales à la condition que $n < e^2$ (ou, en d'autre termes, que la matrice d'ajustement est l'identité). Très récemment, Williamson a produit plusieurs contre-exemples à cette conjecture, en utilisant des arguments géométriques, voir [130].

Notons finalement que la première introduction des algèbres de Hecke est dûe à Iwahori [74], dans le but d'étudier les groupes réductifs finis. Si G est un groupe de

Chevalley défini sur un corps fini, et B un sous-groupe de Borel de G, alors il a montré que l'algèbre de Hecke associée au groupe de Weyl sous-jacent est isomorphe à l'algèbre d'endomorphismes $\mathrm{End}_G(\mathbb{C}[G/B])$.

Représentations de $G(l, 1, n)$ et son algèbre de Hecke

Les motivations étant posées, il est naturel d'essayer de généraliser ces notions et résultats à des groupes plus généraux que le groupe symétrique. En particulier, $\mathfrak{S}_n$ est classifié comme un *groupe de réflexions complexes*, c'est-à-dire engendré par des éléments fixant point par point un hyperplan d'un $\mathbb{C}$-espace vectoriel. La classification de ces groupes a été faite en 1954 par Shephard et Todd dans leur article [125]. Le groupe $\mathfrak{S}_n$ est alors vu comme le groupe $G(l, p, n)$, avec $l = p = 1$, comme l'explique plus en détails le chapitre 2 de ce mémoire.

Dans la suite, on s'intéresse au cas du groupe $W_n = G(l, 1, n)$, qui présente l'avantage de pouvoir être construit comme produit semi-direct de $\mathfrak{S}_n$ et de n copies du groupe cyclique d'ordre l (c'est donc un produit en couronne). Ses représentations ordinaires sont alors indexées par les l-uplets de partitions de somme totale n, cf. paragraphe 2.1.3.

On peut alors définir l'algèbre de Hecke de W_n, appelée dans ce cas algèbre d'Ariki-Koike (voir [7]), et notée $\mathbf{H}_n$. Elle est définie comme une déformation de l'algèbre du groupe W_n, et dépend de paramètres $u, V_1, \dots, V_l$. Il est également intéressant de voir qu'elle peut être définie comme un quotient de l'algèbre de Hecke affine de type A (voir par exemple [117]). A l'instar de $\mathcal{H}_n$ et $\mathfrak{S}_n$, sa théorie des représentations est liée à celle de W_n. Si le cas ordinaire est bien compris, la théorie modulaire de $\mathbf{H}_n$ est, encore une fois, beaucoup plus compliquée.

Parmi les résultats important, un critère de semi-simplicité est donné par d'Ariki [1], voir aussi le théorème 2.2.9 tiré de [35]. Un des problèmes nouveaux est qu'il y a plusieurs manières naturelles d'indexer les représentations irréductibles de $\mathbf{H}_n$ dans le cas modulaire, à savoir :

1. par les l-partitions *de Kleshchev*, comme l'ont montré Ariki et Mathas [8], voir aussi [4],
2. par les l-partitions *FLOTW*, voir l'article de Jacon [75],
3. par les l-partitions *d'Uglov*, comme exposé dans le livre de Geck et Jacon [53].

En ce qui concerne la règle de branchement modulaire pour ces algèbres, elle est donnée en 2007 par Ariki [5]. L'article de Mathas [117] passe en revue une bonne partie de la théorie des algèbres d'Ariki-Koike.

On peut définir de même une matrice de décomposition pour $\mathbf{H}_n$, notée $D(\mathbf{H}_n)$, pour laquelle les questions 1 sont à nouveau posées :

Questions 2.

1. Est-il possible de trouver une forme particulière pour $D(\mathbf{H}_n)$?
2. Comment la calculer explicitement ?

La théorie des ensembles basiques pour $\mathbf{H}_n$ permet de traiter le premier point. La réponse à la deuxième question est donnée par le théorème d'Ariki (théorème 3.3.1), qui généralise la conjecture de Lascoux, Leclerc et Thibon [95]. Celle-ci utilise la théorie des représentations de l'algèbre quantique affine $\mathcal{U}_q(\widehat{\mathfrak{sl}_e})$.

Enfin, il reste à mentionner la théorie des *algèbres de Cherednik rationelles*, qui sont des algèbres de réflexions symplectiques associées aux groupes de réflexion complexes, introduites pour la première fois par Etingof et Ginzburg dans [37]. Dans le cas de W_n, elles sont appelées algèbres de Cherednik *cyclotomiques*, et notées $H_{c,n}$, (elles dépendent en particulier d'un paramètre c explicité au paragraphe 2.3.1).

La théorie des représentations particulièrement riche de $H_{c,n}$ est notamment reliée à celle de $\mathbf{H}_n$, via notamment l'existence du foncteur KZ, voir par exemple [124] ou [60]. On peut également citer les liens avec :

- la géométrie algébrique via les schémas de Hilbert, voir [61] et [62].
- les q-algèbres de Schur, comme développé par Rouquier dans [123],
- les opérateurs de Dunkl, voir par exemple l'article de revue [38], ou celui de Rouquier [121].

L'article de Gordon [59] présente également quelques liens avec ces différents domaines.

L'algèbre quantique affine de type A

Les groupes quantiques : introduction par la physique mathématique

Au début des années 1980, dans le cadre des systèmes intégrable quantiques, notamment à travers l'étude des problèmes de diffusion inverse par Fadeev, de nouvelles structures algébriques sont apparues, dont les généralisations ont ensuite été appelées groupes quantiques dans les travaux de Drinfel'd [36] et Jimbo [81]. Ils ont montré que ce sont des algèbres de Hopf[1] pouvant être vues, dans la plupart des cas, comme des déformations des algèbres universelles enveloppantes des algèbres de Kac-Moody à l'aide d'un paramètre q, jouant un rôle analogue à la constante de Planck en mécanique quantique. En prenant la limite $q \to 1$, on retombe sur la structure classique de l'algèbre universelle enveloppante.

Un des enjeux premiers était la détermination de solutions de l'équation de Yang-Baxter, dont une écriture universelle est

$$(R\otimes\mathbf{1})(\mathbf{1}\otimes R)(R\otimes\mathbf{1}) = (\mathbf{1}\otimes R)(R\otimes\mathbf{1})(\mathbf{1}\otimes R),$$

où R est une matrice agissant sur deux objets parmi trois. Les solutions de cette équation permettent en fait de décrire les relations de commutation définissant l'algèbre quantique associée. Il se trouve que ces équations de Yang-Baxter sont omniprésentes en

[1] En raison de cette structure d'algèbre, on priviliégiera dans cette thèse la terminologie *algèbre quantique* plutôt que celle, pourtant plus standard, de *groupe quantique* popularisée par Drinfel'd.

mécanique statistique ainsi qu'en théorie quantique des champs (en particulier dans la théorie conforme). Ainsi, la théorie des groupes quantiques et de ces équations constitue un paradigme important en physique mathématique.

En physique théorique moderne, la notion de symétrie et d'invariance joue un rôle important. En fait, et malgré leur définition comme déformations des algèbres usuelles, les algèbres quantiques possèdent plusieurs propriétés leur permettant d'être vus comme les "groupes de symétries" associés aux systèmes intégrables quantiques. Plus précisément, dans le modèle XXZ de Heisenberg, le Hamiltonien possède des propriétés d'invariance par le groupe quantique $\mathcal{U}_q(\widehat{\mathfrak{sl}_2})$ dans le cas où le nombre de sites de la chaîne de spin tend vers l'infini (et q générique), ce qui permet de le diagonaliser. Ce fut notamment l'objet des travaux de Jimbo dans les années 1980. Le cas d'une chaîne de spin finie a été compris plus récemment : l'invariance est obtenue à l'aide du même groupe quantique, mais où q est spécialisé en une racine de l'unité.

Au-delà de ces applications en physique théorique, la théorie des groupes quantiques trouve un écho dans de nombreux domaines des mathématiques. Si une partie en est utilisée dans cette thèse pour étudier la théorie des représentations modulaires des groupes de réflexions (et des algèbres de Hecke), on peut en outre citer des applications en théorie des :

- invariants de nœuds et entrelacs,
- invariants de 3-variétés,
- représentations des algèbres de Kac-Moody en caractéristique positive,
- représentations géométrique,
- C^*-algèbres.

Pour des généralités concernant les groupes quantiques et leurs applications, on peut se référer par exemple au livre de Chari et Pressley [24].

Lien avec le groupe symétrique

En 1995, Lascoux, Leclerc et Thibon [95] remarquent que la règle de branchement modulaire étudiée par Kleshchev [93] donne lieu à un graphe, dont la construction combinatoire coïncide avec celle du *graphe cristallin* d'un certain module $V(\Lambda)$ pour l'algèbre $\mathcal{U}_q(\widehat{\mathfrak{sl}_e})$ explicité par Misra et Miwa dans [118]. En s'intéressant à la matrice de la base canonique de $V(\Lambda)$, ils conjecturent que sa spécialisation en 1 donne les matrices de décomposition de $\mathcal{H}_n$, l'algèbre de Hecke de $\mathfrak{S}_n$. Ils donnent également un algorithme permettant le calcul explicite de cette matrice de base canonique. Peu après, Ariki démontre dans [2] la version générale de cette conjecture (en niveau l quelconque), permettant le calcul de la matrice de décomposition de l'algèbre de Hecke $\mathcal{H}_n$ de $W_n = G(l,1,n)$. Ce sera le premier signe du lien particulier entre la théorie des représentations modulaires de W_n et les représentations de l'algèbre quantique affine de type A.

Dans cette optique, la représentation de Fock $\mathcal{F}_{\mathbf{s}}$ de niveau l (où $\mathbf{s}$ est un paramètre dans $\mathbb{Z}^l$) joue un rôle essentiel. Il s'agit d'une représentation de $\mathcal{U}_q(\widehat{\mathfrak{sl}_e})$, de dimension infinie, dont les éléments de la base sont les l-partitions, et permettant donc un traitement combinatoire des représentations de $\mathcal{U}_q(\widehat{\mathfrak{sl}_e})$. Son introduction, dans le cas $l = 1$, est dûe à Jimbo et Miwa [83] et était originellement destinée à étudier un problème de physique mathématique : la théorie des solitons. L'aspect plus algébrique fut ensuite étudié par Misra et Miwa [118], voir aussi [66] et [82]. Les caractéristiques importantes de ce $\mathcal{U}_q(\widehat{\mathfrak{sl}_e})$-module particulier sont l'existence d'une base cristalline, d'un graphe cristallin, et d'une base canonique, comme définis par Kashiwara [86]. Il est à noter qu'en niveau supérieur, l'action de $\mathcal{U}_q(\widehat{\mathfrak{sl}_e})$ sur $\mathcal{F}_{\mathbf{s}}$ nécessite un ordre sur les boîtes des l-partitions, et qu'il existe plusieurs ordres convenables généralisant l'ordre dit "de Hayashi" en niveau 1 (en référence aux travaux [66]), en l'occurence deux. Le premier, utilisé notamment par Ariki, permet la réalisation de $\mathcal{F}_{\mathbf{s}}$ comme produit tensoriel d'espaces de Fock de niveau 1. On parlera alors de réalisation *de Kleshchev*. Le deuxième, introduit par Jimbo, Misra, Miwa et Okado [82], utilisé dans [39] et [126] (dans une version légèrement généralisée) en particulier, est celui que l'on privilégiera dans ce mémoire, cf. sa définition au paragraphe 3.2.1. On dira dans ce cas qu'on a affaire à la réalisation *d'Uglov* de $\mathcal{F}_{\mathbf{s}}$. L' ordre de Kleshchev pourra alors être vu comme une version "asymptotique" de l'ordre d'Uglov, voir la proposition 4.4.1.

Dès lors, outre les auteurs cités plus haut, de nombreux travaux ont mis en évidence les liens entre les deux théories, parmi lesquels on peut citer ceux de Grojnowski [64], Uglov [126], Brundan et Kleshchev [20], ou encore Chuang et Rouquier [30]. En particulier, Uglov a défini dans [126] une base canonique pour l'espace de Fock tout entier. Ses résultats sont exposés en partie 3.2.3. Il se trouve que la matrice de la base canonique de $\mathcal{F}_{\mathbf{s}}$ peut alors être vue comme matrice de décomposition d'une q-algèbre de Schur cyclotomique, voir [127] ou encore [117]. On pourra se référer par exemple à la thèse de Yvonne [131, Chapitre 7] pour des résultats détaillés à ce sujet. De plus, Uglov [126] a donné un algorithme pour le calcul explicite de ces bases canoniques (et donc de la matrice de décomposition de $\mathcal{H}_n$). Un autre algorithme, plus efficace, est exposé dans [76].

Notion de catégorification

Dans leur article [30], Chuang et Rouquier ont défini la notion de $\mathfrak{sl}_2$-catégorification, ensuite généralisée à celle de $\widehat{\mathfrak{sl}_e}$-catégorification par Rouquier dans [122]. Cette nouvelle approche permet alors de donner un cadre dans lequel s'inscrivent tous les travaux des auteurs précédemment cités : la catégorification du module irréductible de plus haut poids $V(\Lambda)$, comme sous-module de $\mathcal{F}_{\mathbf{s}}$.

En fait, Shan [124] a montré que la catégorie $\mathcal{O}$ pour les algèbres de Cherednik cyclotomiques (voir sa définition au paragraphe 2.3.2) induit une catégorification du $\mathcal{U}_q(\widehat{\mathfrak{sl}_e})$-module $\mathcal{F}_{\mathbf{s}}$ tout entier : en le munissant d'opérateurs de i-induction et i-restriction adéquats, elle confère au groupe de Grothendieck de cette catégorie une structure de cristal isomorphe à celle de l'espace de Fock. Losev a ensuite montré qu'avec la réalisation d'Uglov de $\mathcal{F}_{\mathbf{s}}$ (rappelons que seule celle-ci sera utilisée dans cette thèse), les deux graphes ne sont pas seulement isomorphes, mais égaux [107].

Théorie graduée et algèbres de Hecke-carquois

En fait, le théorème de catégorification d'Ariki s'inscrit dans le cadre plus général de la théorie des représentations graduée du groupe symétrique. Récemment, une nouvelle famille d'algèbres a été introduite indépendamment par Rouquier [122], et Khovanov et Lauda [91], les *algèbres de Hecke-carquois* R_d. Parfois également appelées algèbres de Khovanov-Lauda-Rouquier, ou *KLR*, elles sont associées à une algèbre de Kac-Moody $\mathfrak{g}$. De plus, elles ont la particularité d'être canoniquement graduées, et donnent une catégorification de $\mathcal{U}_q^-(\mathfrak{g})$.

En type A, ces algèbres sont Morita équivalentes aux algèbres de Hecke affines. Cependant, au contraire de ces dernières, les algèbres de Hecke-carquois présentent l'avantage d'être canoniquement graduées. En outre, tout comme l'algèbre de Hecke $\mathbf{H}_n$ de W_n peut être obtenue comme quotient de l'algèbre de Hecke affine, il existe un quotient dit *cyclotomique* R_d^Λ de R_d, isomorphe à $\mathbf{H}_n$, voir [21]. Ceci donne lieu à une théorie graduée des représentations de W_n, et on a en particulier une notion de modules de Specht gradués (voir [23]) et de matrice de décomposition graduée. Un de résultats principaux est une version graduée du théorème d'Ariki démontrée par Brundan et Kleshchev [22], qui ont exhibé un isomorphisme

$$V(\Lambda) \xrightarrow{\sim} \bigoplus_{d\geq 0} [\mathrm{Proj}(R_d^\Lambda)]_{\mathbb{Q}(q)}$$

envoyant la base canonique de $V(\Lambda)$ sur une "bonne" base de $[\mathrm{Proj}(R_d^\Lambda)]_{\mathbb{Q}(q)}$ (le groupe de Grothendieck des modules gradués projectifs). Plus de résultats concernant la théorie des représentations graduées de W_n peuvent être trouvés dans [22], ou encore [29].

Plus de théorie des représentations des groupes quantiques

Les groupes quantiques sont aussi étudiés indépendamment de leur lien avec les représentations modulaires de W_n. Dans cette thèse, on utilise seulement une petite partie de la théorie des représentations des groupes quantiques, à savoir le cas où $\mathfrak{g} = \widehat{\mathfrak{sl}_e}$, et on ne s'intéresse qu'aux modules irréductibles de plus haut poids, qui sont de dimension infinie. La théorie des représentations de dimension finie est plus difficile et fait également l'objet de recherche intensives. Elle possède notamment des interactions avec la théorie des algèbres amassées, comme le montrent les travaux de Hernandez et Leclerc [69] quand $\mathfrak{g}$ est de type affine simplement lacé.

On peut également noter que le processus d'affinisation d'une algèbre quantique de type fini peut également être appliqué aux algèbres quantiques affine. L'algèbre finalement obtenue est alors appelée algèbre toroïdale quantique. Leur introduction est dûe à Ginzburg, Kapranov et Vasserot [58] (pour le type A), et à Jing [84] (pour le cas général). Au-delà des représentations de dimension infinie de ces algèbres, les questions concernant le cas de la dimension finie sont source de résultats intéressants, voir par exemple [67] ou [68].

Liens avec d'autres domaines des mathématiques

Combinatoire des mots

Il se trouve que la théorie des cristaux de Kashiwara donne un cadre naturel à la correspondence de Robinson-Schensted-Knuth, qui permet de décrire une relation d'équivalence particulière, comme expliqué au chapitre 5. Cette relation est également décrite de façon purement combinatoire, via les relations de Knuth sur les mots d'un alphabet adapté. Elles définissent alors une structure particulière, appelée *monoïde plaxique* par Lascoux et Schützenberger dans leur article [97], voir aussi le livre [108]. Notons que l'introduction de ce monoïde plaxique était destinée à donner une démonstration parlante de la règle de Littlewood-Richardson pour la multiplication de deux fonctions de Schur.

Depuis, ces structures ont été généralisées à d'autres types, et de nouvelles relations plaxiques ont pu être explicitées, voir par exemple l'article de Littelmann [106] et les travaux de Lecouvey [99], [100] ou encore [101]. Dans le cas du type A affine, on peut généraliser la notion de tableaux de Young semi-standards en utilisant les symboles FLOTW, et utiliser la théorie des cristaux pour obtenir un analogue affine de la correspondance de Robinson-Schensted-Knuth, cf. chapitre 5.

Marches aléatoires et chemins de Littelmann

Comme mentionné ci-dessus, Littelmann [106] a défini une structure plaxique associée à une algèbre de Lie semi-simple quelconque. Pour cela, il utilise le modèle des chemins, définis à partir des *opérateurs de Littelmann*, et donnant lieu au *module de Littelmann*. Ce module s'identifie alors au cristal de Kashiwara.

En fait, cette théorie des chemins de Littelmann possède une interprétation en termes de marches aléatoires, via la transformée de Pitman, et initiée dans les travaux d'O'Connell [119] et [120]. Par la suite, Lecouvey, Lesigne et Peigné ont étudié les marches aléatoires conditionnés dans la continuité de ceux d'O'Connell, voir [103], [104], [102], pour obtenir des comportements asymptotiques de multiplicités tensorielles, illustrant le lien entre théorie des représentations des algèbres de Lie et probabilités.

Citons aussi les travaux de Biane, Bougerol et O'Connell [11], [12], qui ont généralisé cette interprétation pour introduire une transformée de Pitman sur certains mouvements Browniens, vus comme des limites de marches aléatoires centrées. Dans cette optique, ils définissent une version continue des opérateurs de Littelmann, donnant lieu à un "cristal continu", dont la théorie est développée dans [12].

Représentations des groupes unitaires et conjectures de Hiss

Récemment, des connexions entre théorie des caractères des groupes unitaires et cristaux d'espaces de Fock semblent apparaître. Considérons le groupe unitaire $G = GU_n(p)$ où p est une puissance d'un nombre premier, et ses caractères unipotents, voir par exemple [49]. Ils sont paramétrés par les partitions de n. En choisissant alors un nombre premier d divisant l'ordre de G ne divisant pas p, on peut étudier les représentations d-modulaires de G.

On peut en particulier définir une matrice de décomposition, carrée, dont les lignes sont indexées par les caractères ordinaires, et les colonnes par les caractères d-modulaires. Chacun de ces ensembles se décompose en *séries de Harish-Chandra*. Dans le cas ordinaire, il existe une caractérisation combinatoire simple et très satisfaisante des différentes séries de Harish-Chandra : elles sont déterminées par le 2-cœur des partitions, [109], [40]. Le cas modulaire est encore une fois plus compliqué, comme le montrent les travaux de Geck, Hiss et Malle [49], [50].

Dans ce cas, l'induction de Harish-Chandra permet de définir un graphe orienté sur certains modules unipotents dont les sommets sources sont les modules cuspidaux. Des expérimentations conduisent à conjecturer que ces graphes apparaissent comme composantes du graphe cristallin d'un espace de Fock de niveau 2 bien choisi. Dans un projet en commun avec G. Hiss, nous envisageons de réinvestir les résultats de cette thèse (notamment du chapitre 5) dans l'étude de cette conjecture.

Présentation des résultats

Comme expliqué ci-avant, cette thèse en combinatoire algébrique est à l'interface de la théorie des représentations modulaires du groupe symétrique $\mathfrak{S}_n$ d'une part (et plus généralement du groupe de réflexions $G(l,1,n)$), et de la théorie des représentations de l'algèbre affine quantique $\mathcal{U}_q(\widehat{\mathfrak{sl}_e})$ (via notamment la structure de cristal induite par les représentations de Fock) d'autre part.

Un des problèmes originels était une généralisation d'un théorème de Geck et Jacon [53, Théorème 6.7.2] concernant les ensembles basiques pour les algèbre d'Ariki-Koike non semi-simples $\mathbf{H}_{k,n}$ (paramétrées par $\mathbf{s} \in \mathbb{Z}^l$ et $e \in \mathbb{Z}_{>1}$). Rappelons que la notion d'ensemble basique formalise le fait d'avoir une matrice de décomposition unitriangulaire. Après avoir défini un ordre $\ll_{\mathbf{m}}$ sur les l-partitions chargées dépendant d'un paramètre $\mathbf{m} \in \mathbb{Q}^l$, Geck et Jacon montrent qu'il existe une valeur particulière de $\mathbf{m}$ telle que l'algèbre d'Ariki-Koike $\mathbf{H}_{k,n}$ admette un ensemble basique par rapport à l'ordre $\ll_{\mathbf{m}}$, à savoir un ensemble de l-partitions d'Uglov associées à $\mathbf{s}$ et e.

On peut alors se demander si pour $\mathbf{m}$ fixé, on peut toujours trouver un ensemble basique pour $\mathbf{H}_{k,n}$ par rapport à $\ll_{\mathbf{m}}$. Sinon, comment classifier entièrement quels paramètres donnent lieu à des ordres permettant d'avoir un ensemble basique ? Et dans le cas où $\mathbf{m}$ induit un ensemble basique, peut-on le décrire explicitement ?

Le chapitre 4 de cette thèse donne une réponse constructive à ces questions. On définit d'abord un ensemble fini $\mathscr{P}^*$ d'hyperplans de $\mathbb{Q}^l$, dépendant de $\mathbf{s}$ et de e. On montre alors le théorème de classification suivant (théorème 4.5.2) :

- Si $\mathbf{m} \notin \mathscr{P}^*$, alors $\mathbf{H}_{k,n}$ admet un ensemble basique par rapport à l'ordre $\ll_{\mathbf{m}}$, à savoir un ensemble de l-partitions d'Uglov *tordues* associées à $\mathbf{r}$ et e, et où $\mathbf{r}$ est la l-charge $\mathbf{m}$-*adaptée*. Dans un cas particulier, dit *asymptotique*, cet ensemble coïncide avec l'ensemble des l-partitions de *Kleshchev*.
- Si $\mathbf{m} \in \mathscr{P}^*$, alors $\mathbf{H}_{k,n}$ ne possède pas d'ensemble basique par rapport à $\ll_{\mathbf{m}}$.

Pour en arriver à cette conclusion, il est important de comprendre la combinatoire sous-jacente des symboles associés aux l-partitions chargées. On montre alors qu'on peut réaliser l'espace de Fock de manière alternative, en tordant l'action des générateurs de $\mathcal{U}_q(\widehat{\mathfrak{sl}_e})$. On applique ensuite le théorème d'Ariki pour en déduire un paramétrage alternatif des modules irréductibles dans les cas non semi-simples, qui formeront l'ensemble basique recherché.

Ces ensemble basiques nouvellement déterminés et jusqu'alors inconnus, viennent donc compléter la liste des ensemble basiques déterminés au préalable pour les algèbres de Ariki-Koike dans le livre [53].

Un deuxième aspect des travaux de recherche exposés dans ce mémoire est l'étude plus approfondie, au chapitre 5, des graphes cristallins des espaces de Fock, notion essentielle à la fois pour déterminer la base canonique de Kashiwara afin d'appliquer le théorème d'Ariki, mais également pour comprendre la structure de cristal sur la catégorie des modules simples pour l'algèbre de Cherednik (grâce aux théorèmes de Shan [124] et Losev [107]).

Par la théorie de Kashiwara, on sait qu'il existe des isomorphismes de cristaux entre composantes connexes de différents espaces de Fock. On peut ainsi définir une relation d'équivalence entre l-partitions : $\boldsymbol{\lambda} \sim \boldsymbol{\mu}$ si leurs graphes cristallins sont isomorphes et si $\boldsymbol{\lambda}$ et $\boldsymbol{\mu}$ aparaissent au même endroit dans leur cristal respectif. Un ensemble de représentants particuliers de cette classe d'équivalence est donné par les partitions FLOTW, qui correspondent à une composante connexe de plus haut poids vide. Comment expliciter alors cette relation d'équivalence, et, partant de $\boldsymbol{\lambda} \in \mathcal{F}_{\mathbf{s}}$ quelconque, déterminer simplement la l-partition FLOTW $\boldsymbol{\mu} \in \mathcal{F}_{\mathbf{r}}$ telle que $\boldsymbol{\lambda} \sim \boldsymbol{\mu}$?

En s'inspirant des résultats similaires dans le cas des l-partitions de plus haut poids exposé dans [78], on répond à la question en construisant l'*isomorphisme de cristaux canonique* à partir de trois isomorphismes élémentaires :

- le premier directement inspiré de l'algorithme d'insertion de Schensted, pour les symboles,
- l'isomorphisme de *cyclage* caractéristique du cas affine, qui peut être vu comme un analogue affine du cyclage défini par Lascoux et Schützenberger [108]
- un nouvel isomorphisme, dit de *réduction*, qui supprime des parts de même taille dans une l-partition cylindrique.

On souligne le fait que cet isomorphisme canonique peut alors être vu comme une version affine de l'algorithme d'insertion de Schensted. De plus, on peut le raffiner en bijection, ce qui donne à la fois :

1. une version affine de la correspondance de Robinson-Schensted-Knuth,
2. une description non récursive, jusqu'ici inconnue, des l-partitions constituant une composante connexe quelconque.

Les objets construits dans le cas affine peuvent alors, pour certains, être mis en parallèle avec ceux déjà connus dans le cas du type A fini (voir par exemple le livre [73] pour le cas fini), comme le montre le tableau ci-après. Les notations sont celles du chapitre 5. Notons que les différents algorithmes construits dans le cas affine ont été implémentés en Python.

<table>
<tr><th></th><th>Type A fini</th><th>Type A affine</th></tr>
<tr><td>Algèbre quantique</td><td>$\mathcal{U}_q(\mathfrak{sl}_e)$</td><td>$\mathcal{U}_q(\widehat{\mathfrak{sl}_e})$</td></tr>
<tr><td>Espace de Fock</td><td>Base = multipartitions chargées par $\mathbf{s}$ avec $s_i \in [\![1, e-1]\!]$</td><td>Base = multipartitions chargées par $\mathbf{s}$ / $\mathbf{s}$-symboles</td></tr>
<tr><td>Module irréductible de plus haut poids</td><td>$\mathcal{U}_q(\mathfrak{sl}_e).|\boldsymbol{\emptyset}, \mathbf{s}\rangle$</td><td>$\mathcal{U}_q(\widehat{\mathfrak{sl}_e}).|\boldsymbol{\emptyset}, \mathbf{s}\rangle$</td></tr>
<tr><td>Graphe cristallin</td><td>Sommets = multipartitions de profondeur inférieure à e</td><td>Sommets = symboles FLOTW (si $\mathbf{s} \in \mathscr{S}_e^l$)</td></tr>
<tr><td rowspan="3">Isomorphismes de cristaux</td><td>RS pour les tableaux</td><td>RS pour les symboles</td></tr>
<tr><td></td><td>cyclage ξ</td></tr>
<tr><td></td><td>réduction ρ</td></tr>
<tr><td>Relation d'équivalence recherchée</td><td>$\mathbf{T}_1 \sim \mathbf{T}_2$ ssi $\mathbf{RS}(\mathbf{T}_1) = \mathbf{RS}(\mathbf{T}_2)$</td><td>$\boldsymbol{\lambda}_1 \sim \boldsymbol{\lambda}_2$ ssi $\Phi(\boldsymbol{\lambda}_1) = \Phi(\boldsymbol{\lambda}_2)$ où Φ est l'isomorphisme de cristaux canonique</td></tr>
</table>

Développements possibles

Les résultats démontrés dans ce mémoire, ainsi que les techniques utilisées, permettent d'envisager plusieurs directions dans lesquelles orienter de futurs travaux de recherche. La partie "perspectives", située à la fin de cette thèse, explique plus en détail de quoi il retourne. Mentionnons pour l'instant les quelques points suivants.

1. Dans la lignée du chapitre 4 et l'étude approfondie des matrices de décomposition pour les algèbres d'Ariki-Koike, on peut se poser la question suivante : Est-il possible d'avoir deux lignes égales dans une telle matrice de décomposition ? Dans son article [129], Wildon traite le cas $l = 1$, et répond à cette question négativement, sauf dans le cas $e = 2$ où les cas d'égalité sont explicités. Notons que si on connaît la réponse dans le cas des algèbres d'Ariki-Koike, on la connaît également pour les algèbres de groupes associées, en sachant que la matrice d'ajustement entre les deux matrices de décomposition est unitriangulaire (voir [115, Théorème 6.35]).

2. Le chapitre 5 donne la construction d'un algorithme analogue à celui de Robinson-Schensted-Knuth pour le cas affine. Il est connu que cette correspondance induit une structure de monoïde sur les mots, décrite par les relations plaxiques, voir [108]. Peut-on obtenir des relations similaires pour décrire la correspondance dans le cas affine ?

3. On a évoqué précédemment le lien entre cristaux, chemins de Littelmann, et transformée de Pitman. Peut-on, dans le cas du type A affine, utiliser les isomorphismes de cristaux du chapitre 5 pour étendre ces liens, et déterminer la transformée de Pitman généralisée de façon explicite dans l'esprit du travail originel d'O'Connell ?

4. Dans quelle mesure les représentations des groupes unitaires, en particulier la caractérisation des séries de Harish-Chandra, peuvent-elle être comprises à travers la théorie des cristaux de Kashiwara et les isomorphismes déterminés dans cette thèse ? Une réponse conjecturale est apportée par les récentes observations de Hiss [70].

Chapter 1

Preliminaries

This preliminary chapter consits of four parts. In the first section, we recall some basic definitions and facts about the combinatorics of multipartitions, which are the underlying objects when it comes to the representation theory of the complex reflection group $G(l,1,n)$, as exposed in Chapter 2. Then, we define some orders on the set of multipartitions and investigate how they are related. The third part is concerned with the extended affine symmetric group, and its action on $\mathbb{Z}^l$. Finally, we introduce a particular set of multipartitions called the FLOTW multipartitions, which will be useful throughout this thesis.

1.1 Combinatorics of multipartitions

1.1.1 Charged multipartitions

We start by some classic definitions.

Partition of an integer

Let $n \in \mathbb{Z}_{\geq 0}$. A *partition* of n is a sequence of integers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_p)$, for some $p \in \mathbb{Z}_{\geq 0}$, verifying $\lambda_i \geq \lambda_{i+1} > 0$ for all $i \in [\![1, p-1]\!]$, and $\sum_{i=1}^p \lambda_i = n$. We say that n is the *rank* of n and write $n = |\lambda|$. The elements λ_i, $i \in [\![1, p]\!]$ are called the *parts* of λ. The integer p is called the *height* of λ. The set of partitions of n is denoted by $\Pi(n)$. Moreover, we write $\lambda \vdash n$ if $\lambda \in \Pi(n)$. We will sometimes consider that a partition λ of n has, in addition, an infinite number of empty parts, and write $\lambda = (\lambda_1, \dots, \lambda_p, \lambda_{p+1}, \dots)$, where $\lambda_i = 0$ if and only if $i \geq p+1$. Also, we use the multiplicative notation for partitions, e.g. $(4, 2, 2, 1, 0, \dots) = (4.2^2.1)$. The *empty partition*, denoted $\emptyset$, is the partition of 0.

Charged multipartitions

Fix now $l \in \mathbb{Z}_{>0}$. An *l-partition* (also called a *multipartition* when there is no need to specify the value of l) is an l-tuple of partitions $\boldsymbol{\lambda} = (\lambda^1, \dots, \lambda^l)$ such that $\sum_{c=1}^l =$

$|\lambda^c| = n$. Again, we call n the *rank* of $\boldsymbol{\lambda}$ and denote it by $|\boldsymbol{\lambda}|$; the parts of the different partitions λ^c are called the *parts* of $\boldsymbol{\lambda}$; and the *l-height* of $\boldsymbol{\lambda}$ is $h(\boldsymbol{\lambda}) = (h(\lambda^1), \dots, h(\lambda^l))$. The set of l-partitions of rank n is denoted by $\Pi_l(n)$, and we write $\boldsymbol{\lambda} \vdash_l n$ if $\boldsymbol{\lambda} \in \Pi_l(n)$. The *empty multipartition*, denoted $\boldsymbol{\emptyset}$, is the l-partition $(\emptyset, \dots, \emptyset)$.

Example 1.1.1. $(3.2, 4.1^3, \emptyset, 2^2)$ is a 4-partition of 16.

An element $\mathbf{s} = (s_1, s_2, \dots, s_l) \in \mathbb{Z}^l$ is called an *l-charge* (or simply a *multicharge*). A *charged l-partition* is a formal element denoted $|\boldsymbol{\lambda}, \mathbf{s}\rangle$ where $\boldsymbol{\lambda}$ is an l-partition and $\mathbf{s}$ is an l-charge. The next section justifies the terminology "charge" for $\mathbf{s}$.

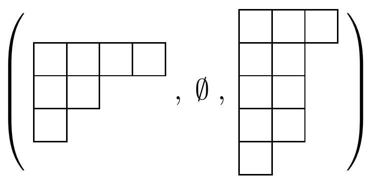
1.1.2 Young diagram with contents representation

Let $\boldsymbol{\lambda} \vdash_l n$. The *Young diagram* of $\boldsymbol{\lambda}$ is the set

$$[\boldsymbol{\lambda}] = \{(a,b,c)\ ;\ a \geq 1,\ c \in [\![1,l]\!],\ b \in [\![1,\lambda^c_a]\!]\}\,.$$

It is depicted by an l-tuple of arrays, where the c-th array contains a box in row a and column b if and only if $(a,b,c) \in [\boldsymbol{\lambda}]$. In other terms, the c-th array is the superposition of $h(\lambda^c)$ rows, where the a-th row consists of λ^c_a boxes.

Example 1.1.2. Let $\boldsymbol{\lambda} = (4.2.1, \emptyset, 3.2^3.1) \vdash_3 17$.

$[\boldsymbol{\lambda}] =$ 

From now on, we will often identify a multipartition $\boldsymbol{\lambda}$ and its Young diagram $[\boldsymbol{\lambda}]$. Each box is labeled by a triple (a,b,c) of $[\boldsymbol{\lambda}]$. These elements are called the *nodes* of $\boldsymbol{\lambda}$. If γ is node of $\boldsymbol{\lambda}$, we say that γ is *removable* (or a node *of type R*) if $[\boldsymbol{\lambda}]\backslash\{\gamma\}$ (that is, the Young diagram of $\boldsymbol{\lambda}$ from which we have removed the box labeled by γ) is still the Young diagram of some $\boldsymbol{\mu} \vdash_l n-1$. In this case, we also say that γ is an *addable* node (or a *node of type A*) of $\boldsymbol{\mu}$.

Example 1.1.3. Let $\boldsymbol{\lambda} = (4.2.1, \emptyset, 3.2^3.1)$, as in the previous example. Then one sees that the removable nodes of $\boldsymbol{\lambda}$ are the nodes

$$(1,4,1), (2,2,1), (3,1,1), (1,3,3), (4,2,3), (5,1,3),$$

and the addable nodes of $\boldsymbol{\lambda}$ are the nodes

$$(1,5,1), (2,3,1), (3,2,1), (4,1,1), (1,1,2), (1,4,3), (2,3,3), (5,2,3), (6,1,3).$$

Fix now an l-charge $\mathbf{s}$, and choose a node $\gamma = (a,b,c) \in [\boldsymbol{\lambda}]$. One defines the *content* of γ as follows :

$$\mathrm{cont}_{\boldsymbol{\lambda}}(\gamma) = b - a + s_c.$$

Also, given $e \in \mathbb{Z}_{>1}$, one defines the *residue* of γ by

$$\mathrm{res}_{\boldsymbol{\lambda}}(\gamma) = \mathrm{cont}_{\boldsymbol{\lambda}}(\gamma) \mod e.$$

We will omit the index in this notation when there is no possible confusion. Moreover, for $i \in [\![0, e-1]\!]$, we call γ an *i-node* if $\mathrm{res}(\gamma) = i$. We further denote $\gamma^+ = (a, b+1, c)$, i.e. the node located to the right of γ; and $\gamma^- = (a, b-1, c)$, i.e. the node located to the left of γ. Of course, $\delta = \gamma^+ \Leftrightarrow \gamma = \delta^-$. Also, if γ is an i-node, then γ^+ is an $(i+1)$-node.

Given a charged l-partition $|\boldsymbol{\lambda}, \mathbf{s}\rangle$, i.e. an l-partition and an l-charge, one can draw the associated Yound diagram whose boxes are filled with the corresponding contents. We then identify $|\boldsymbol{\lambda}, \mathbf{s}\rangle$ and this representation.

Example 1.1.4. Let $\boldsymbol{\lambda} = (1^2, 3, 2.1) \vdash_3 8$ and $\mathbf{s} = (1, -4, 5)$. We write

$$|\boldsymbol{\lambda}, \mathbf{s}\rangle = \left(\begin{array}{|c|}\hline 1 \\ \hline 0 \\ \hline \end{array}\, , \begin{array}{|c|c|c|}\hline -4 & -3 & -2 \\ \hline \end{array}\, , \begin{array}{|c|c}\hline 5 & \multicolumn{1}{c|}{6} \\ \hline 4 & \\ \cline{1-1} \end{array} \right).$$

1.1.3 Shifted symbol representation

There is another, equivalent way, to represent charged multipartitions, namely by using shifted symbols, as originally defined in [15].

Let $\mathbf{m} \in \mathbb{Q}^l$, and let $\boldsymbol{\lambda} = (\lambda^1, \dots, \lambda^l)$ be an l-partition. Let $p \geq \max\limits_{1 \leq c \leq l}(1 + h(\lambda^c) - \lfloor m_c \rfloor)$. For $c \in [\![1, l]\!]$, and for $a \in [\![1, p + \lfloor m_c \rfloor]\!]$, define

$$\mathfrak{B}_a^c(\boldsymbol{\lambda}) = \lambda_a^c - a + p + m_c.$$

The *β-number* of λ^c is the sequence

$$\mathfrak{B}_{\mathbf{m}}^c(\boldsymbol{\lambda}) = (\mathfrak{B}_{p+\lfloor m_c \rfloor}^c(\boldsymbol{\lambda}), \dots, \mathfrak{B}_1^c(\boldsymbol{\lambda})).$$

One can picture the sequence of β-numbers $\mathfrak{B}_{\mathbf{m}}(\boldsymbol{\lambda}) = (\mathfrak{B}_{\mathbf{m}}^1(\boldsymbol{\lambda}), \dots, \mathfrak{B}_{\mathbf{m}}^l(\boldsymbol{\lambda}))$ in an array whose c-th row, numbered from bottom to top, is $\mathfrak{B}_{\mathbf{m}}^c(\boldsymbol{\lambda})$. This array is called the *(shifted)* $\mathbf{m}$*-symbol* of size p of $\boldsymbol{\lambda}$. It has $T(\boldsymbol{\lambda}, \mathbf{m}) = lp + \sum_{1 \leq c \leq l} \lfloor m_c \rfloor$ elements.

Example 1.1.5. Take $\boldsymbol{\lambda} = (1.1, \emptyset, 2) \vdash_3 4$ and $\mathbf{m} = (1/3, 2, -1)$. Choose $p = 3$. The β-numbers are the following:

$$\mathfrak{B}_{\mathbf{m}}^1(\boldsymbol{\lambda}) = (1/3, 7/3, 10/3), \quad \mathfrak{B}_{\mathbf{m}}^2(\boldsymbol{\lambda}) = (0, 1, 2, 3, 4) \quad \text{and} \quad \mathfrak{B}_{\mathbf{m}}^3(\boldsymbol{\lambda}) = (0, 3).$$

The shifted $\mathbf{m}$-symbol of size 3 is then given by

$$\mathfrak{B}_{\mathbf{m}}(\boldsymbol{\lambda}) = \begin{pmatrix} 0 & 3 & & & \\ 0 & 1 & 2 & 3 & 4 \\ 1/3 & 7/3 & 10/3 & & \end{pmatrix}$$

In practice, we can construct $\mathfrak{B}_{\mathbf{m}}(\boldsymbol{\lambda})$ as follows.

1. We start by drawing the array whose c-th row has size $p + \lfloor m_c \rfloor$, and contains the integers from 0 to $p + \lfloor m_c \rfloor$ in increasing order.

2. We add the fractional part $m_c - \lfloor m_c \rfloor$ to each entry of the c-th row of the symbol we have obtained. Note that, at this point, we have constructed the shifted symbol $\mathfrak{B}_{\mathbf{m}}(\boldsymbol{\emptyset})$ of the empty l-partition.

3. For each $c \in [\![1, l]\!]$, we add:

   - the part λ_1^c to the last entry of the c-th row,
   - the part λ_2^c to the last but one entry of the c-th row,
   - and so on...

Example 1.1.6. Let $\boldsymbol{\lambda}$ and $\mathbf{s}$ be as in Example 1.1.5. Then the tree steps above give:

1. $\begin{pmatrix} 0 & 1 & & & \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & & \end{pmatrix}$,

2. $\begin{pmatrix} 0 & 1 & & & \\ 0 & 1 & 2 & 3 & 4 \\ 1/3 & 4/3 & 7/3 & & \end{pmatrix} = \mathfrak{B}_{\mathbf{m}}(\boldsymbol{\emptyset})$,

3. $\begin{pmatrix} 0 & \mathbf{3} & & & \\ 0 & 1 & 2 & 3 & 4 \\ 1/3 & \mathbf{7/3} & \mathbf{10/3} & & \end{pmatrix} = \mathfrak{B}_{\mathbf{m}}(\boldsymbol{\lambda})$, where the bold entries are the ones that have been modified by adding the parts of $\boldsymbol{\lambda}$.

Remark 1.1.7. Note that all steps being invertible, it is clear that starting from a symbol, and provided we know its size, we can recover the corresponding multicharge and the multipartition by doing the inverse manipulations.

In the particular case where $\mathbf{m}$ is a multicharge, say $\mathbf{m} = \mathbf{s} \in \mathbb{Z}^l$, one can compute the $\mathbf{s}$-symbol of $\boldsymbol{\lambda}$ by forgetting the second step, and we see that it is equivalent to either consider $|\boldsymbol{\lambda}, \mathbf{s}\rangle$ or its shifted symbol $\mathfrak{B}_{\mathbf{s}}(\boldsymbol{\lambda})$.

We end this paragraph with a definition which will be useful in Chapter 5.

Definition 1.1.8. Let $|\boldsymbol{\lambda}, \mathbf{s}\rangle$ be a charged l-partition. Then the symbol $\mathfrak{B}_{\mathbf{s}}(\boldsymbol{\lambda})$ is called *semistandard* if

1. $s_1 \leq s_2 \leq \cdots \leq s_l$, and

2. the entries in each column of $\mathfrak{B}_{\mathbf{s}}(\boldsymbol{\lambda})$ are non-decreasing (when read from top to bottom).

Remark 1.1.9. Note that a symbol representing a charged multipartition has, by definition, rows with increasing entries (when read from left to right). Therefore, the definition of the semistandardness for symbols is coherent with the classic one on Young tableaux, see e.g. [42].

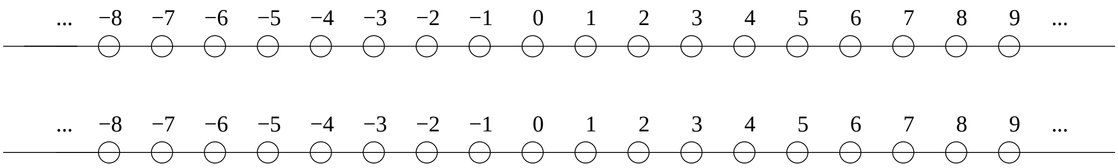
1.1.4 Abacus representation

In [126], Uglov uses another representation of charged l-partitions, very similar to the symbol representation, namely, the abacus representation. Let us recall his construction, since it will help defining an order on multipartition in the following section, see Paragraph 1.2.3. Note that this is also detailed in the thesis of Yvonne [131]. The introduction of abacuses to represent multipartitions is due to James and Kerber [80].

Let $|\boldsymbol{\lambda}, \mathbf{s}\rangle$ be a charged l-partition. We use the running example $l = 2$, $\mathbf{s} = (0, 1)$ and $\boldsymbol{\lambda} = (2.1, \emptyset) \vdash_2 3$. We first start by computing the shifted symbol $\mathfrak{B}_{\mathbf{s}}(\boldsymbol{\lambda})$ of appropriate size p. In our example, we choose $p = 3$, and we get

$$\mathfrak{B}_{\mathbf{s}}(\boldsymbol{\lambda}) = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & \end{pmatrix}.$$

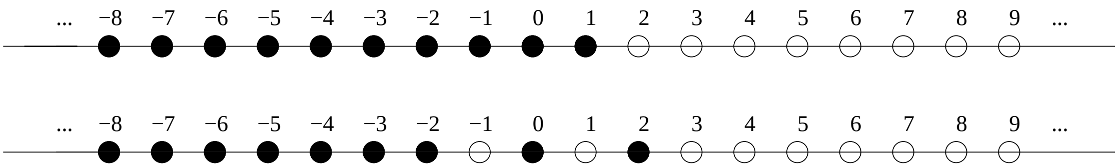
Consider the infinite "empty" l-abacus consisting in l rows (numbered from bottom to top), each of which being indexed by $\mathbb{Z}$ and containing only empty spots, represented by white beads. For $l = 2$, this gives



Then, we define the abacus $\mathcal{A}^l_{\mathbf{s}}(\boldsymbol{\lambda})$ to be the l-abacus containing a black bead in row c and position k if and only if

- $k + p - 1$ appears in the c-th row of $\mathfrak{B}_{\mathbf{s}}(\boldsymbol{\lambda})$, or
- $k + p - 1$ is smaller than any entry in the c-th row of $\mathfrak{B}_{\mathbf{s}}(\boldsymbol{\lambda})$.

Note that the second condition implies that the l-abacus is completely filled on the left. In our case, we get the following 2-abacus:



Remark 1.1.10. By Remark 1.1.7, it is straightforward that the map $|\boldsymbol{\lambda}, \mathbf{s}\rangle \longrightarrow \mathcal{A}^l_{\mathbf{s}}(\boldsymbol{\lambda})$ is a bijection. The inverse map is given as follows. Starting from an l-abacus,

- the c-th component of the multicharge corresponds to the position of the rightmost black bead on the c-th row of the abacus once all black beads have been moved to the left,

- the c-th component of the multipartition is obtained by counting the numbers of white beads appearing to the left of each black bead of the c-th row of the abacus.

As already mentioned above, this representation finds its utility in the order on $\Pi_l(n)$ it induces. More precisely, in Paragraph 1.2.3, we wil use a different indexation of the beads of $\mathcal{A}^l_{\mathbf{s}}(\boldsymbol{\lambda})$, depending on a parameter $e \in \mathbb{Z}_{>1}$, to construct this order.

1.2 Ordering multipartitions

There is a natural way to order partitions, namely using the dominance order. However, when it comes multipartitions, there are several ways to generalise this order. We expose here three different ways to order multipartitions.

1.2.1 Dominance order on partitions and multipartitions

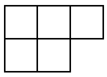
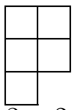
Dominance on partitions

Let us recall the general notion of dominance. Let $\alpha = (\alpha_1, \alpha_2, \dots)$ and $\beta = (\beta_1, \beta_, \dots)$ be two sequences of rational numbers, such that $\sum_i \alpha_i = \sum_i \beta_i$. We say that α *dominates* β, and we write $\alpha \trianglerighteq \beta$, if for all $d \geq 1$,

$$\sum_{1\leq i\leq d} \alpha_i \geq \sum_{1\leq i\leq d} \beta_i.$$

Moreover, we write $\alpha \triangleright \beta$ if $\alpha \trianglerighteq \beta$ and $\lambda \neq \beta$.

This applies in particular to the case where α and β are partitions of the same integer n.

Example 1.2.1. Let $\lambda =$  and $\mu =$ . Then $\lambda_1 = 3 > 2 = \mu_1$, $\lambda_1 + \lambda_2 = 3 + 2 > 2 + 2 = \mu_1 + \mu_2$ and $\lambda_1 + \lambda_2 + \lambda_3 = 3 + 2 + 0 = 2 + 2 + 1 = \mu_1 + \mu_2 + \mu_3$, which proves that $\lambda \triangleright \mu$.

Remark 1.2.2. Note that this is a partial order. For instance, the partitions (4.1^2) and (3^2) are not comparable.

Dominance on multipartitions

Perhaps the most natural way to generalise this dominance order to l-partitions is, for $\boldsymbol{\lambda}, \boldsymbol{\mu} \vdash_l n$, to check whether "$\boldsymbol{\lambda}$ dominates $\boldsymbol{\mu}$ component by component". This is what is formalised in the following definition.

Definition 1.2.3. We say that $\boldsymbol{\lambda} = (\lambda^1, \dots, \lambda^l)$ *dominates* $\boldsymbol{\mu} = (\mu^1, \dots, \mu^l)$ and we write $\boldsymbol{\lambda} \trianglerighteq \boldsymbol{\mu}$ if, for all $c \in [\![1, l]\!]$ and for all $d \geq 1$,

$$\sum_{k=1}^{c-1} |\lambda^c| + \sum_{a=1}^{d} \lambda^c_i \geq \sum_{k=1}^{c-1} |\mu^c| + \sum_{a=1}^{d} \mu^c_i.$$

We further write $\boldsymbol{\lambda} \rhd \boldsymbol{\mu}$ if $\boldsymbol{\lambda} \unrhd \boldsymbol{\mu}$ and $\boldsymbol{\lambda} \neq \boldsymbol{\mu}$.

We can also define a twisted version of this order. For $\pi \in \mathfrak{S}_l$, we write $\boldsymbol{\lambda} \unrhd^{\pi} \boldsymbol{\mu}$ if, for all $c \in [\![1, l]\!]$ and for all $d \geq 1$,

$$\sum_{k=1}^{c-1} |\lambda^{\pi(c)}| + \sum_{a=1}^{d} \lambda_i^{\pi(c)} \geq \sum_{k=1}^{c-1} |\mu^{\pi(c)}| + \sum_{a=1}^{d} \mu_i^{\pi(c)}.$$

Alternatively, we have $\boldsymbol{\lambda} \unrhd^{\pi} \boldsymbol{\mu}$ if and only if $\boldsymbol{\lambda}^{\pi} \unrhd \boldsymbol{\mu}^{\pi}$, where $\boldsymbol{\lambda}^{\pi} = (\lambda^{\pi(1)}, \dots, \lambda^{\pi(l)})$ and $\boldsymbol{\mu}^{\pi} = (\mu^{\pi(1)}, \dots, \mu^{\pi(l)})$.

Remark 1.2.4. Note that the dominance order on $\Pi_l(n)$ does not depend on any parameter. In the upcoming two sections, we define orders on $\Pi_l(n)$ depending on some parameters (notably a multicharge).

1.2.2 Order on shifted symbols

Let $\boldsymbol{\lambda}$ be an l-partition, and let $\mathbf{m} \in \mathbb{Q}^l$. Consider the shifted $\mathbf{m}$-symbol $\mathfrak{B}_{\mathbf{m}}(\boldsymbol{\lambda})$. Write

$$\mathfrak{b}_{\mathbf{m}}(\boldsymbol{\lambda}) = (\mathfrak{b}_{\mathbf{m}}^1(\boldsymbol{\lambda}), \mathfrak{b}_{\mathbf{m}}^2(\boldsymbol{\lambda}), \dots, \mathfrak{b}_{\mathbf{m}}^{T(\boldsymbol{\lambda},\mathbf{m})}(\boldsymbol{\lambda}))$$

the sequence of of elements of $\mathfrak{B}_{\mathbf{m}}(\boldsymbol{\lambda})$ in decreasing order, i.e. $\mathfrak{b}_{\mathbf{m}}^i(\boldsymbol{\lambda}) \leq \mathfrak{b}_{\mathbf{m}}^j(\boldsymbol{\lambda})$ if $i < j$. We further denote

$$n_{\mathbf{m}}(\boldsymbol{\lambda}) = \sum_{i=1}^{T(\boldsymbol{\lambda},\mathbf{m})} (i-1)\mathfrak{b}_{\mathbf{m}}^i(\boldsymbol{\lambda}). \tag{1.1}$$

Definition 1.2.5. For $n \in \mathbb{Z}_{\geq 0}$, one defines the order $\ll_{\mathbf{m}}$ on $\Pi_l(n)$ by

$$\boldsymbol{\lambda} \ll_{\mathbf{m}} \boldsymbol{\mu} \quad \text{if} \quad \boldsymbol{\lambda} = \boldsymbol{\mu} \text{ or } \mathfrak{b}_{\mathbf{m}}(\boldsymbol{\lambda}) \rhd \mathfrak{b}_{\mathbf{m}}(\boldsymbol{\mu}),$$

for all $\boldsymbol{\lambda}, \boldsymbol{\mu} \vdash_l n$.

Note that this is a partial order. We will also consider the particular case where $\mathbf{m}$ is actually a multicharge $\mathbf{s} \in \mathbb{Z}^l$.

Example 1.2.6. Take $\mathbf{m} = \mathbf{s} = (3, 1)$, $\boldsymbol{\lambda} = (\emptyset, 3)$, $\boldsymbol{\mu} = (2.1, 1)$ and $\boldsymbol{\nu} = (1^3, \emptyset)$. Then the associated $\mathbf{s}$-symbols (of size 4) are

$$\mathfrak{B}_{\mathbf{s}}(\boldsymbol{\lambda}) = \begin{pmatrix} 0 & 4 & & \\ 0 & 1 & 2 & 3 \end{pmatrix},$$

$$\mathfrak{B}_{\mathbf{s}}(\boldsymbol{\mu}) = \begin{pmatrix} 0 & 2 & & \\ 0 & 1 & 3 & 5 \end{pmatrix},$$

$$\mathfrak{B}_{\mathbf{s}}(\boldsymbol{\nu}) = \begin{pmatrix} 0 & 1 & & \\ 0 & 2 & 3 & 4 \end{pmatrix},$$

which yields

$$\mathfrak{b}_{\mathbf{s}}(\boldsymbol{\lambda}) = (4,3,2,1,0,0), \quad \mathfrak{b}_{\mathbf{s}}(\boldsymbol{\mu}) = (5,3,1,1,0,0) \quad \text{and} \quad \mathfrak{b}_{\mathbf{s}}(\boldsymbol{\nu}) = (4,3,2,1,0,0).$$

We see that $\boldsymbol{\mu} \ll_{\mathbf{s}} \boldsymbol{\lambda}$ and $\boldsymbol{\mu} \ll_{\mathbf{s}} \boldsymbol{\nu}$, but that $\boldsymbol{\lambda}$ and $\boldsymbol{\nu}$ are not comparable with respect to $\ll_{\mathbf{s}}$.

Remark 1.2.7. This order depends on the parameter $\mathbf{m}$ (which is a multicharge when $\mathbf{m}$ is actually an element of $\mathbb{Z}^l$.)

1.2.3 Uglov's order

Using the abacus representation of l-partitions, we define another order on the set of l-partitions. Fix $n \in \mathbb{Z}_{>0}$, and introduce a parameter $e \in \mathbb{Z}_{>1}$. For a charged l-partition $|\boldsymbol{\lambda}, \mathbf{s}\rangle$, consider its corresponding l-abacus $\mathcal{A}^l_{\mathbf{s}}(\boldsymbol{\lambda})$. We carry on using the same running example as in Paragraph 1.1.4, namely $\mathbf{s} = (0, 1)$ and $\boldsymbol{\lambda} = (2.1, \emptyset)$. Also, we choose $e = 3$.

Divide $\mathcal{A}^l_{\mathbf{s}}(\boldsymbol{\lambda})$ into rectangles of width e and height l, indexed by $k \in \mathbb{Z}$, and such that the k-th rectangle contains all beads indexed by $(k-1)e+1, (k-1)e+2, \dots, ke-1, ke$. In our example, this gives

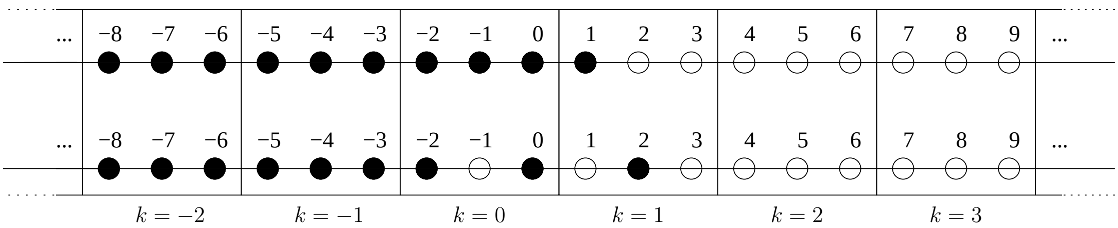


Figure 1.1: The abacus $\mathcal{A}^l_{\mathbf{s}}(\boldsymbol{\lambda})$.

Now, we change the indexation of the beads, so that the beads in the k-th rectangle are indexed by $(k-1)el + 1, \dots, kel - 1, kel$, from left to right, starting from the first (i.e. bottom) row. For instance, the previous abacus becomes

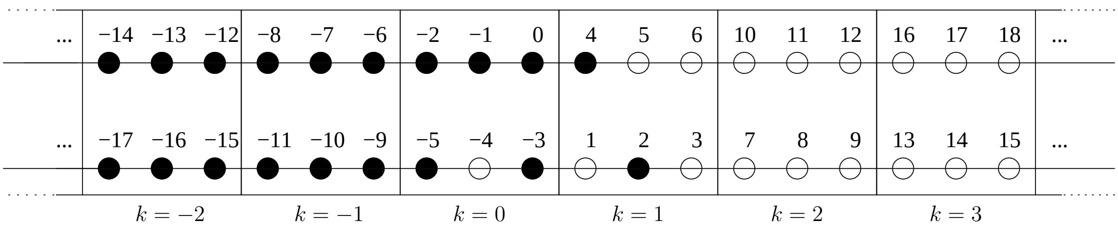


Figure 1.2: The abacus $\mathcal{A}^l_{\mathbf{s}}(\boldsymbol{\lambda})$ with modified indexation of the beads.

The set of all beads in the l-abacus is then indexed by $\mathbb{Z}$. We can therefore construct a 1-abacus, by simply reordering the numbered beads on one single row. Here, we get the following 1-abacus.

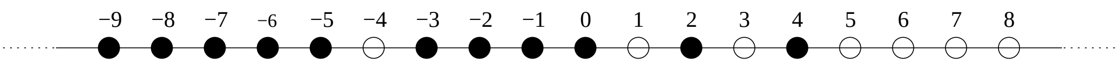


Figure 1.3: The abacus $\mathcal{A}^1_{\tau(\mathbf{s})}(\tau(\boldsymbol{\lambda}))$.

This abacus represents a charged partition, which we denote by $|\tau(\boldsymbol{\lambda}), \tau(\mathbf{s})\rangle$. We can compute $\tau(\mathbf{s})$ and $\tau(\boldsymbol{\lambda})$ following the procedure in Remark 1.1.10. In this example, we find $\tau(\boldsymbol{\lambda}) = (3.2.1^4) \vdash 9$ and $\tau(\mathbf{s}) = 1$. One has the relation

$$\tau(\mathbf{s}) = s_1 + \cdots + s_l.$$

Definition 1.2.8. Let $e \in \mathbb{Z}_{>1}$, and $n \in \mathbb{Z}_{>0}$. We define the order $\leq_{\mathcal{U}}$ on $\Pi_l(n)$ by

$$\boldsymbol{\lambda} \leq_{\mathcal{U}} \boldsymbol{\mu} \quad \text{if} \quad \tau(\boldsymbol{\lambda}) \trianglelefteq \tau(\boldsymbol{\mu}),$$

for all $\boldsymbol{\lambda}, \boldsymbol{\mu} \vdash_l n$, in the usual sense of dominance for partitions.

Remark 1.2.9. Again, this is a partial order. Besides, $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are comparable only if $\tau(\boldsymbol{\lambda})$ and $\tau(\boldsymbol{\mu})$ are partitions of the same integer.

Remark 1.2.10. This order depends on two parameters: the multicharge $\mathbf{s}$ (as does the order $\ll_{\mathbf{s}}$ in the previous section), and the integer e.

1.2.4 Compatibiliy between the different orders

We now give a compatibility property between the orders defined in 1.2.5 and 1.2.8.

Proposition 1.2.11. *Let $\mathbf{s} \in \mathbb{Z}^l, e \in \mathbb{Z}_{>1}$ and $n \in \mathbb{Z}_{\geq 0}$. Suppose that there exists $\pi \in \mathfrak{S}_l$ such that $c < c' \Rightarrow s_{\pi(c)} - s_{\pi(c')} > n + 2e - 1$. Then*

$$\boldsymbol{\mu} \leq_{\mathcal{U}} \boldsymbol{\lambda} \Rightarrow \boldsymbol{\lambda} \ll_{\mathbf{s}} \boldsymbol{\mu}$$

for all $\boldsymbol{\lambda}, \boldsymbol{\mu} \vdash_l n$.

Proof. We show that in this case, both orders are compatible with the π-twisted dominance order on l-partitions $\trianglerighteq^{\pi}$ (see Definition 1.2.3). Remark that the three orders defined above all rely ultimately on the dominance order of partitions. Roughly speaking, the condition on $\mathbf{s}$ ensures that there is no interaction between the rows of the abacus (respectively of the symbol). We illustrate this in Example 1.2.12 below. Note that it is sufficient to prove it in the case $l = 2$ (since in the general case, π decomposes as a product of transpositions). Therefore, we only consider the case where $\pi = \mathrm{Id}$ or $\pi = (12)$.

More precisely, let us first look at Uglov's order on abacuses $\leq_{\mathcal{U}}$. For all $\boldsymbol{\lambda} = (\lambda^1, \lambda^2) \vdash_2 n$, consider the abacus representation of $|\boldsymbol{\lambda}, \mathbf{s}\rangle$. Because of the particular shape of the multicharge $\mathbf{s}$, there is necessarily a rectangle whose $\pi(1)$-th row has only black beads, and whose $\pi(2)$-th row has only white beads. Let k be the index of this rectangle. Therefore, when constructing the corresponding 1-abacus $\mathcal{A}^1_{\tau(\mathbf{s})}(\tau(\boldsymbol{\lambda}))$, the beads indexed by integers smaller than $(k-1)el + 1$ represent to the partition $\lambda^{\pi(2)}$, and the ones indexed by integers greater than kel represent the partition $\lambda^{\pi(1)}$. In other terms, the partition $\tau(\boldsymbol{\lambda})$ can be read directly on the 2-abacus. Because the π-twisted dominance order on l-partitions is just dominance with twisted lexicographic convention, we have in fact

$$\boldsymbol{\mu} \leq_{\mathcal{U}} \boldsymbol{\lambda} \Leftrightarrow \tau(\boldsymbol{\mu}) \trianglelefteq \tau(\boldsymbol{\lambda}) \Rightarrow \boldsymbol{\mu} \trianglelefteq^{\pi} \boldsymbol{\lambda}.$$

Note that here we do not have an equivalence since $\boldsymbol{\mu}$ and $\boldsymbol{\lambda}$ can be comparable with respect to $\trianglelefteq^{\pi}$ but uncomparable with respect to $\leq_{\mathcal{U}}$: this can happen when $|\tau(\boldsymbol{\lambda})| \neq |\tau(\boldsymbol{\mu})|$.

As to the order on symbols, we have the same kind of phenomenon. When constructing the sequence $\mathfrak{b}_{\mathbf{s}}(\boldsymbol{\lambda})$, there is an entry $\alpha = \mathfrak{B}_a^c(\boldsymbol{\lambda})$ for some a and c such that the entries smaller than α in $\mathfrak{b}_{\mathbf{s}}(\boldsymbol{\lambda})$ represent to the partition $\lambda^{\pi(2)}$, and such that the entries greater than α represent the partition $\lambda^{\pi(1)}$. Looking at the dominance order (in the sense of partitions) on the sequences $\mathfrak{b}_{\mathbf{s}}(\boldsymbol{\lambda})$ therefore comes down to looking at the π-twisted dominance order on the 2-partitions $\boldsymbol{\lambda}$, i.e.

$$\boldsymbol{\lambda} \ll_{\mathbf{s}} \boldsymbol{\mu} \Leftrightarrow \boldsymbol{\lambda} \trianglelefteq^{\pi} \boldsymbol{\mu},$$

for all $\boldsymbol{\lambda}, \boldsymbol{\mu} \vdash_l n$.

This gives the expected compatibiliy property. □

Example 1.2.12. Take $l = 2$, $n = 3$, $e = 3$, $\mathbf{s} = (0, 8)$. Consider $\boldsymbol{\lambda} = (\emptyset, 3)$ and $\boldsymbol{\mu} = (2, 1)$. Then the associated 2-abacuses are

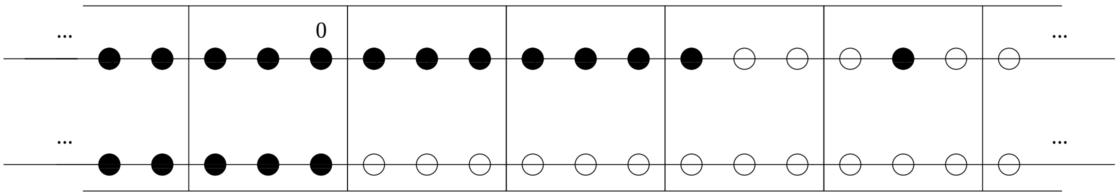


Figure 1.4: The abacus $\mathcal{A}^2_{\mathbf{s}}(\boldsymbol{\lambda})$.

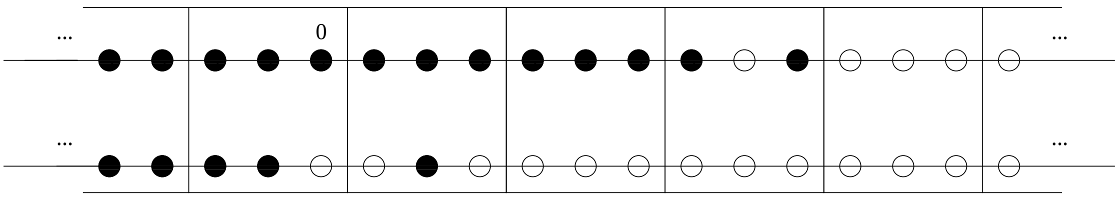


Figure 1.5: The abacus $\mathcal{A}^2_{\mathbf{s}}(\boldsymbol{\mu})$.

Following Uglov's procedure, we get the following corresponding 1-abacuses

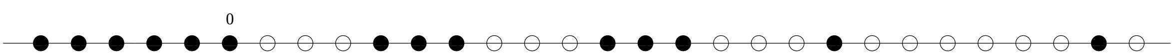


Figure 1.6: The abacus $\mathcal{A}^1_{\tau(\mathbf{s})}(\tau(\boldsymbol{\lambda}))$.

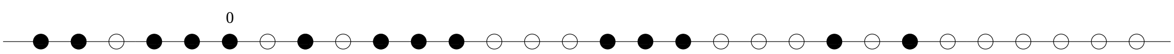


Figure 1.7: The abacus $\mathcal{A}^1_{\tau(\mathbf{s})}(\tau(\boldsymbol{\mu}))$.

We deduce that $\tau(\mathbf{s}) = 8$, $\tau(\boldsymbol{\lambda}) = (15.9.6^3.3^3) \vdash 51$ and $\tau(\boldsymbol{\mu}) = (10.9.6^3.3^3.2.1^3) \vdash 51$. We have $\tau(\boldsymbol{\lambda}) \rhd \tau(\boldsymbol{\mu})$, thus $\boldsymbol{\mu} \leq_{\mathcal{U}} \boldsymbol{\lambda}$.

On the other hand, we have

$$\mathfrak{B}_{\mathbf{s}}(\boldsymbol{\lambda}) = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 15 \\ 0 & 1 & 2 & 3 & & & & & & & & \end{pmatrix}$$

$$\mathfrak{B}_{\mathbf{s}}(\boldsymbol{\mu}) = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 12 \\ 0 & 1 & 2 & 5 & & & & & & & & \end{pmatrix}$$

This gives

$$\mathfrak{b}_{\mathbf{s}}(\boldsymbol{\lambda}) = (15, 10, 9, 8, 7, 6, 5, 4, 3, 3, 2, 2, 1, 1, 0, 0)$$

$$\mathfrak{b}_{\mathbf{s}}(\boldsymbol{\mu}) = (12, 10, 9, 8, 7, 6, 5, 5, 4, 3, 2, 2, 1, 1, 0, 0)$$

We have $\mathfrak{b}_{\mathbf{s}}(\boldsymbol{\lambda}) \rhd \mathfrak{b}_{\mathbf{s}}(\boldsymbol{\mu})$, hence $\boldsymbol{\lambda} \ll_{\mathbf{s}} \boldsymbol{\mu}$.

We also see that we have $\boldsymbol{\mu} \lhd^{(12)} \boldsymbol{\lambda}$.

1.3 The extended affine symmetric group

In this paragraph, we define a particular group, acting on $\mathbb{Z}^l$ (where $l \in \mathbb{Z}_{>0}$). It turns out that the Ariki-Koike algebra defined in Section 2.2.1, which depends on a parameter $\mathbf{s} \in \mathbb{Z}^l$, is invariant in the orbits with respect to this action.

1.3.1 Two presentations of $\widehat{\mathfrak{S}_l}$

First presentation

Let $l \in \mathbb{Z}_{>0}$. The *extended affine symmetric group* $\widehat{\mathfrak{S}_l}$ is the group with the following presentation:

- Generators: σ_i, $i \in [\![1, l-1]\!]$ and y_i, $i \in [\![1, l]\!]$.
- Relations :
  - $\sigma_i^2 = 1$ for all $i \in [\![1, l-1]\!]$,
  - $\sigma_i \sigma_{i+1} \sigma = \sigma_{i+1} \sigma_i \sigma_{i+1}$ for all $i \in [\![1, l-2]\!]$,
  - $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $i, j \in [\![1, l-1]\!]$ such that $|i-j| > 1$,
  - $y_i y_j = y_j y_i$ for all $i, j \in [\![1, l]\!]$,
  - $\sigma_i y_j = y_j \sigma_i$ for $i \in [\![1, l-1]\!]$ and $j \in [\![1, l]\!]$ such that $j \neq i, i+1 \mod l$,
  - $\sigma_i y_i \sigma_i = y_{i+1}$ for all $i \in [\![1, l-1]\!]$.

Note that this is not a Coxeter group. Also, $\widehat{\mathfrak{S}_l}$ can be regarded as the semidirect product $\mathbb{Z}^l \rtimes \mathfrak{S}_l$. Indeed, one can identify

- the parabolic subgroup generated by the elements σ_i, for $i \in [\![1, l-1]\!]$, with the symmetric group $\mathfrak{S}_l$ (where σ_i is the transposition $(i, i+1)$), and
- the parabolic subgroup generated by the elements y_i, for $i \in [\![1, l]\!]$, with the group $\mathbb{Z}^l$ (where y_i is the element of the standard basis of $\mathbb{Z}^l$, containing 1 in position i and 0 elsewhere).

Second presentation

Equivalently, $\widehat{\mathfrak{S}_l}$ is characterised by the following presentation:

- Generators: σ_i, $i \in [\![0, l-1]\!]$ and τ.
- Relations :
  - $\sigma_i^2 = 1$ for all $i \in [\![0, l-1]\!]$,
  - $\sigma_i \sigma_{i+1} \sigma = \sigma_{i+1} \sigma_i \sigma_{i+1}$ for all $i \in [\![0, l-1]\!]$,
  - $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $i, j \in [\![0, l-1]\!]$ such that $i - j \neq 1 \mod l$,
  - $\tau \sigma_i = \sigma_{i+1} \tau$ for all $i \in [\![0, l-1]\!]$,

  where the indices are read modulo l in the second and last formulas.

With this presentation, one can regard $\widehat{\mathfrak{S}_l}$ as the semidirect product $\widetilde{\mathfrak{S}_l} \rtimes \langle \tau \rangle$, where $\widetilde{\mathfrak{S}_l}$ is the affine symmetric group. Indeed, $\widetilde{\mathfrak{S}_l}$ is precisely the parabolic subgroup generated by the elements σ_i, for $i \in [\![0, l-1]\!]$.

1.3.2 Action of $\widehat{\mathfrak{S}_l}$ on $\mathbb{Z}^l$

Definition of the action

We consider the first presentation of $\widehat{\mathfrak{S}_l}$. Fix an integer $e \in \mathbb{Z}_{>1}$. Then there is an action of $\widehat{\mathfrak{S}_l}$ on the set of multicharges $\mathbb{Z}^l$ via the following formulas:

- $\sigma_i.\mathbf{s} = (s_1, \dots, s_{i-1}, s_{i+1}, s_i, \dots, s_l)$ for all $i \in [\![1, l-1]\!]$, and
- $y_i.\mathbf{s} = (s_1, \dots, s_{i-1}, s_i + e, s_{i+1}, \dots, s_l)$ for all $i \in [\![1, l]\!]$,

for all $\mathbf{s} = (s_1, \dots, s_l) \in \mathbb{Z}^l$.

In other terms, the generator σ_i permutes the i-th and the $(i+1)$-th components of $\mathbf{s}$, and the generator y_i translates the i-th component of $\mathbf{s}$ by e.

A fundamental domain

It is then easy to see that the set

$$\mathscr{D}_e^l = \Big\{(s_1, \dots, s_l) \in \mathbb{Z}^l \,|\, 0 \leq s_1 \leq \dots \leq s_l < e\Big\} \tag{1.2}$$

is a fundamental domain for this action.

We also define another particular subset of $\mathbb{Z}^l$ which will be of great importance in the rest of this thesis. We set

$$\mathscr{S}_e^l = \Big\{(s_1, \dots, s_l) \in \mathbb{Z}^l \mid 0 \leq s_c - s_{c'} < e \quad \text{for } c < c'\Big\}. \tag{1.3}$$

In particular, we have $\mathscr{D}_e^l \subset \mathscr{S}_e^l$.

Orbits

For $\mathbf{s} \in \mathbb{Z}^l$, denote

- $\mathcal{C}(\mathbf{s})$ the orbit of $\mathbf{s}$ under the action of $\widehat{\mathfrak{S}_l}$, and
- $\mathcal{C}_e(\mathbf{s})$ the orbit of $\mathbf{s}$ under the action of the parabolic subgroup $\langle y_1, \dots, y_l \rangle$ of $\widehat{\mathfrak{S}_l}$.

Note that $\mathbf{r} \in \mathcal{C}_e(\mathbf{s})$ if and only if $\mathbf{r}$ is obtained from $\mathbf{s}$ by translating each of the components by some multiple of e.

Example 1.3.1. Let $\mathbf{s} = (2, 3, -4)$ and $e = 3$. Set also $\mathbf{r} = (8, 3, -1)$ and $\mathbf{t} = (-7, 2, 9)$. Then $\mathbf{r} = (s_1 + 2e, s_2, s_3 + e)$, thus $\mathbf{r} \in \mathcal{C}_e(\mathbf{s})$; and $\mathbf{t} = (s_3 - e, s_1, s_2 + 2e)$, thus $\mathbf{t} \in \mathcal{C}(\mathbf{s}) \backslash \mathcal{C}_e(\mathbf{s})$. The representative of $\mathbf{s}$ in $\mathscr{D}_e^l$ is $(0, 2, 2) = (s_3 - e, s_2, s_1 + 2e)$.

1.4 FLOTW multipartitions

We end this first chapter by the introduction of a particular subset of $\Pi_l(n)$, the set of FLOTW multipartitions. They are named after Foda, Leclerc, Okado, Thibon and Welsh, who introduced them in [39]. They have a nice explicit combinatorial definition, as stated in the definition below.

1.4.1 Definition

As in the previous section, we set $l \in \mathbb{Z}_{>0}$ and $e \in \mathbb{Z}_{>1}$.

Definition 1.4.1. Let $\boldsymbol{\lambda} = (\lambda^1, \dots, \lambda^l)$ be an l-partition and $\mathbf{s} \in \mathscr{S}_e^l$. Then $\boldsymbol{\lambda}$ is called an *FLOTW l-partition* if:

1. For all $c \in [\![1, l-1]\!]$, $\lambda_a^c \geq \lambda_{a+s_{c+1}-s_c}^{c+1}, \forall a \geq 1$; and $\lambda_a^l \geq \lambda_{a+e+s_1-s_l}^1, \forall a \geq 1$.
2. For all $\alpha > 0$, the residues of the rightmost nodes of the parts of size α do not cover $[\![0, e-1]\!]$.

Denote by $\Psi_{\mathbf{s}}$ the set of FLOTW l-partitions associated to $\mathbf{s} \in \mathscr{S}_e^l$, and by $\Psi_{\mathbf{s}}(n) \subset \Psi_{\mathbf{s}}$ the ones of rank n, for $n \in \mathbb{Z}_{\geq 0}$.

The l-partitions verifying only the first condition are called *cylindric*.

Remark 1.4.2. Suppose that $l = 1$. Then the first condition vanishes, and the second condition is equivalent to saying that there is no sequence of e parts of the same size in $\boldsymbol{\lambda}$. Indeed, since in this case $\boldsymbol{\lambda}$ is a partition, all parts of a given size α are consecutive. Thus, the contents of their rightmost nodes are consecutive. Their residues therefore cover $[\![0, e-1]\!]$ if and only if there are at least e parts of size α. But this is exactly the definition of an e-regular partition (see e.g. [80] for more details). This means that the FLOTW multipartitions for $l = 1$ are the e-regular partitions.

In fact, FLOTW l-partitions have been introduced to generalise the notion of e-regular partitions to $l > 1$: these partitions are known to label the irreducible e-modular representation of $\mathfrak{S}_n$ (cf. [80]); and FLOTW l-partitions label the irreducible e-modular representations of the generalisation $G(l,1,n)$ of $\mathfrak{S}_n$, as explained in [39, Section 3.4] (see also Theorem 4.2.1 in Chapter 4).

1.4.2 Cyclage and symbol characterisation of cylindric multipartitions

Let $|\boldsymbol{\lambda}, \mathbf{s}\rangle$ be a charged l-partition. Write $\boldsymbol{\lambda} = (\lambda^1, \dots, \lambda^l)$ and $\mathbf{s} = (s_1, \dots, s_l)$.

Definition 1.4.3. The *cyclage* of $|\boldsymbol{\lambda}, \mathbf{s}\rangle$ is the charged l-partition $|\xi(\boldsymbol{\lambda}), \xi(\mathbf{s})\rangle$ defined by

- $\xi(\mathbf{s}) = (s_l - e, s_1, \dots, s_{l-1})$, and
- $\xi(\boldsymbol{\lambda}) = (\lambda^l, \lambda^1, \dots, \lambda^{l-1})$.

With this tool, we get a more visual description of the cylindric multipartitions (i.e. multipartition verifying only the first point in Definition 1.4.1).

Proposition 1.4.4. *Let $\mathbf{s} \in \mathscr{S}_e^l$. A charged l-partition $|\boldsymbol{\lambda}, \mathbf{s}\rangle$ is cylindric if and only if*

1. *$\mathfrak{B}_{\mathbf{s}}(\boldsymbol{\lambda})$ is semistandard, and*
2. *$\mathfrak{B}_{\xi(\mathbf{s})}(\xi(\boldsymbol{\lambda}))$ is semistandard.*

Proof. In this proof, we will denote $\xi(\boldsymbol{\lambda})_i$ (resp. $\xi(\mathbf{s})_i$) the i-th component of $\xi(\boldsymbol{\lambda})$ (resp. $\xi(\mathbf{s})$). We have

$$\begin{aligned}
\boldsymbol{\lambda}_a^c \geq \boldsymbol{\lambda}_{a+s_{c+1}-s_c}^{c+1} \quad &\Leftrightarrow \quad \boldsymbol{\lambda}_a^c - a + p + s_c \geq \boldsymbol{\lambda}_{a+s_{c+1}-s_c}^{c+1} - a + p + s_c \\
&\Leftrightarrow \quad \boldsymbol{\lambda}_a^c - a + p + s_c \geq \boldsymbol{\lambda}_{a+s_{c+1}-s_c}^{c+1} - (a + s_{c+1} - s_c) + p + s_{c+1} \\
&\Leftrightarrow \quad \boldsymbol{\lambda}_a^c - a + s_c \geq \boldsymbol{\lambda}_{a+s_{c+1}-s_c}^{c+1} - (a + s_{c+1} - s_c) + s_{c+1} \\
&\Leftrightarrow \quad \mathfrak{B}_a^c(\boldsymbol{\lambda}) \geq \mathfrak{B}_{a+s_{c+1}-s_c}^{c+1}(\boldsymbol{\lambda}) \qquad \text{by definition of those numbers.}
\end{aligned}$$

Besides,

$$\begin{aligned}
\boldsymbol{\lambda}^l_a \geq \boldsymbol{\lambda}^1_{a+e+s_1-s_l} \quad &\Leftrightarrow \quad \xi(\boldsymbol{\lambda})^1_a \geq \xi(\boldsymbol{\lambda})^2_{a+e+s_1-s_l} \\
&\Leftrightarrow \quad \xi(\boldsymbol{\lambda})^1_a - a - e + p + s_l \geq \xi(\boldsymbol{\lambda})^2_{a+e+s_1-s_l} - a - e + p + s_l \\
&\Leftrightarrow \quad \xi(\boldsymbol{\lambda})^1_a - a - e + p + s_l \geq \xi(\boldsymbol{\lambda})^2_{a+e+s_1-s_l} - (a + e + s_1 - s_l) + p + s_1 \\
&\Leftrightarrow \quad \xi(\boldsymbol{\lambda})^1_a - a + \xi(\mathbf{s})_1 \geq \xi(\boldsymbol{\lambda})^2_{a+\xi(\mathbf{s})_2-\xi(\mathbf{s})_1} - (a + \xi(\mathbf{s})_2 - \xi(\mathbf{s})_1) + \xi(\mathbf{s})_2 \\
&\Leftrightarrow \quad \mathfrak{B}^1_a(\xi(\boldsymbol{\lambda})) \geq \mathfrak{B}^2_{a+\xi(\mathbf{s})_2-\xi(\mathbf{s})_1}(\xi(\boldsymbol{\lambda})).
\end{aligned}$$

Now, this is exactly what we want, because

- in the symbol representation of $|\boldsymbol{\lambda}, \mathbf{s}\rangle$, the entry just above $\mathfrak{B}^c_a(\boldsymbol{\lambda})$ is $\mathfrak{B}^{c+1}_{a+s_{c+1}-s_c}(\boldsymbol{\lambda})$,
- in the symbol representation of $|\xi(\boldsymbol{\lambda}), \xi(\mathbf{s})\rangle$, the entry just above $\mathfrak{B}^1_a(\boldsymbol{\lambda})$ is $\mathfrak{B}^2_{a+\xi(\mathbf{s})_2-\xi(\mathbf{s})_1}(\xi(\boldsymbol{\lambda}))$.

Note that here, we have used the fact that if $\mathfrak{B}_{\mathbf{s}}(\boldsymbol{\lambda})$ is semistandard, then it is sufficient to check that $\mathfrak{B}_{\xi(\mathbf{s})}(\xi(\boldsymbol{\lambda}))$ is semistandard on the first two rows (the bottom ones) to show that the whole cycled symbol is semistandard.

$\square$

Chapter 2

The complex reflection group $G(l, 1, n)$

In their work [125], Shephard and Todd have fully classified finite groups generated by complex reflections (that is, non-trivial elements fixing a complex hyperplane pointwise). They showed the existence of a single infinite family of groups, depending on three parameters $l, p, n \in \mathbb{Z}_{>0}$, and denoted by $G(l, p, n)$, and of exactly 34 other exceptional groups. A good reference on the subject is the book of Broué [14].

In this thesis, we are interested in a particular case of this infinite series, namely when $p = 1$. In this case, much is known about both the ordinary and modular representation theory of the group, and of its associated objects. This chapter recalls some background about $G(l, 1, n)$, then focuses on its associated Hecke algebra (called the Ariki-Koike algebra), which is the main object of interest in the rest of this thesis. We also give some elements of the representation theory of the rational Cherednik algebra associated with $G(l, 1, n)$, which has been of blossoming interest since its introduction by Etingof and Ginzburg in [37].

2.1 Basic facts about $G(l, 1, n)$

In this section and from now on, we will sometimes write W_n for the complex reflection group $G(l, 1, n)$. We will define $G(l, 1, n)$ in an abstract way, and then explain how it is seen as a group of matrices.

2.1.1 Definition and presentation

Let l, n be two positive integer.

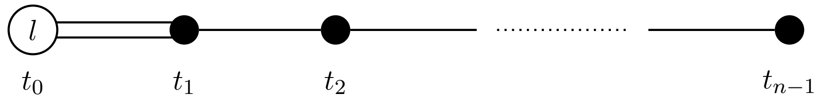
Definition 2.1.1. The complex reflection group $G(l, 1, n)$ is the group

$$W_n = (\mathbb{Z}/l\mathbb{Z})^n \rtimes \mathfrak{S}_n.$$

It has the following presentation:

- generators: t_i for $i \in [\![0, n-1]\!]$,
- relations:
  - $t_0^l = 1$,
  - $t_i^2 = 1$ for $i \geq 1$,
  - $t_0t_1t_0t_1 = t_1t_0t_1t_0$,
  - $t_it_{i+1}t_i = t_{i+1}t_it_{i+1}$ for $i \in [\![1, n-2]\!]$,
  - $t_it_j = t_jt_i$ if $|i-j| > 1$.

These relations are represented by the following "Coxeter-like" diagram:



Remark 2.1.2. In general, this is not a Coxeter group, since t_0 is not an involution. However,

1. If $l = 1$, then t_0 is not a generator anymore, and the diagram reduces to the Coxeter diagram of type A_{n-1}. In other words, $W_n = \mathfrak{S}_n$ (which is also easy to see using the definition of W_n).
2. If $l = 2$, then t_0 is an involution, and W_n is the Coxeter group of type B_n.

Moreover, one sees that $|G| = l^n n!$.

2.1.2 Matrix representation

Actually, reflections being geometrical objects, it is natural to think of this group as a group of matrices, acting on a vector space $V \simeq \mathbb{C}^n$.

In fact,

- t_0 is represented by a diagonal matrix of the form $\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \zeta \end{pmatrix}$ where ζ is a primitive root of unity of order l.
- t_i is represented by the permutation matrix associated to the transposition $(i, i+1)$.

Then, W_n is seen as the subset of $\mathcal{M}_n(\mathbb{C})$ with exactly one non-zero coefficient a_i $(0 \leq i \leq n)$ in each row and each column, verifying $(\prod_{i=1}^n a_i)^l = 1$.

Denote $\mathcal{S}_n$ the set of complex reflections of W_n. For $s \in \mathcal{S}_n$, there is a corresponding reflection hyperplane H_s. Then, choose $\alpha_s \in V^*$ a linear form such that $\ker(\alpha_s) = H_s$, and v_s such that $\mathrm{Span}(v_s)\oplus H_s = V$ and $\alpha_s(v_s) = 1-\det(s)^{-1}$. We also use the notation $\langle x, y\rangle$ for $y(x)$ ($x \in V^*$, $y \in V$).

Let us now give an explicit description of $\mathcal{S}_n$. Take $\zeta = \exp(2\pi\sqrt{-1}/l)$, and write $\{z_1, \dots, z_n\}$ for the standard basis of $V = \mathbb{C}^n$. For $1 \leq i, j, k \leq n$ distinct, let ε_k and t_{ij} be the following elements of $GL(V)$:

$$\begin{aligned} \varepsilon_k(z_k) &= \zeta z_k \quad \text{and} \quad \varepsilon_k(z_i) = z_i, \\ t_{ij}(z_i) &= z_j \quad \text{and} \quad t_{ij}(z_k) = z_k. \end{aligned}$$

In other terms, ε_k is the diagonal matrix containing ε in row and column k, and 1 elsewhere; and t_{ij} is the permutation matrix associated to the transposition (i, j). Then

$$\mathcal{S}_n = \bigsqcup_{p=1}^{l-1} \{\varepsilon_k^p \ ; \ 1 \leq k \leq n\} \sqcup \left\{t_{ij}\varepsilon_i^p\varepsilon_j^{-p} \ ; \ 1 \leq i < j \leq n\right\}. \tag{2.1}$$

2.1.3 Parametrisation of the ordinary representations

We want to parametrise the irreducible representations of W_n over $\mathbb{C}$. It is known that the irreducible representations of $\mathfrak{S}_n$ over $\mathbb{C}$ are parametrised by partitions. Denote by $\mathscr{E}^\lambda$ the irreducible $\mathbb{C}\mathfrak{S}_n$-module (also called Specht module) associated to the partition λ of n.

Now, let $\boldsymbol{\lambda} = (\lambda^1, \dots, \lambda^l)$ be an l-partition of n such that λ^i is a partition of some $n_i \in \mathbb{Z}_{\geq 0}$, $\sum_{i=1}^l n_i = n$. Following the arguments of [113], see also [123, Section 6], one can associate to $\boldsymbol{\lambda}$ the $\mathbb{C}W_n$-module

$$\mathscr{E}^{\boldsymbol{\lambda}} = \mathrm{Ind}_{W_{n_1}\times\cdots\times W_{n_l}}^{W_n}(\mathscr{E}^{\lambda^1} \otimes (\mathscr{E}^{\lambda^2} \otimes \sigma_2) \otimes \cdots \otimes (\mathscr{E}^{\lambda^l} \otimes \sigma_l)),$$

where σ_i is the representation of W_n determined by

$$\begin{array}{rccc} \sigma_i : & t_0 & \longmapsto & \zeta^{i-1} \\ & t_k & \longmapsto & 1, \end{array}$$

and $\zeta = \exp(2\pi\sqrt{-1}/l)$.

Actually, this gives an complete set of non-isomorphic, irreducible $\mathbb{C}W_n$-modules, whence the following parametrisation:

$$\mathrm{Irr}_{\mathbb{C}}(W_n) = \left\{\mathscr{E}^{\boldsymbol{\lambda}} \ ; \ \boldsymbol{\lambda} \vdash_l n\right\}.$$

2.2 Ariki-Koike algebras

This section recalls some well-known facts about the Ariki-Koike algebra, which is just the Hecke algebra associated to W_n, and whose representation theory is similar to that of W_n in the semisimple case. In fact, this algebra is the object of main interest in this thesis, and one original aim was a generalisation of the results in 2.2.3.

The definition of an Ariki-Koike algebra requires the introduction of $l+1$ parameters. We will first consider that these parameters are indeterminates, which will give rise to a "generic", semisimple algebra. Then, every Ariki-Koike algebra will be seen as a "specialisation" of this generic algebra, that is by assigning a value to each of these parameters.

2.2.1 Definition and first properties

Definition as a Hecke algebra

As in Section 2.1, we take $l, n \in \mathbb{Z}_{>0}$. Let $u, V_1, \dots, V_l$ be independent indeterminates. Take $R = \mathbb{Z}[\zeta]$, where $\zeta = \exp(2\pi\sqrt{-1}/l)$ [1] , and set $A = \mathbb{C}[u^{\pm 1}, V_1, \dots, V_l]$.

Definition 2.2.1. The generic Ariki-Koike algebra $\mathbf{H}_n = \mathbf{H}_{A,n}(u, V_1, \dots, V_l)$ is the unital associative A-algebra with generators T_i, $i = 0, \dots, n-1$, and relations

- $(T_i - u)(T_i + 1) = 0$ for all $i \in [\![1, n-1]\!]$.
- $(T_0 - V_1) \dots (T_0 - V_l) = 0$
- $T_0 T_1 T_0 T_1 = T_1 T_0 T_1 T_0$
- $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$ for all $i \in [\![1, n-2]\!]$
- $T_i T_j = T_j T_i$ whenever $|i - j| > 1$

Remark 2.2.2. If one takes $u = 1$ and $V_i = 1$ for all $i = 1, \dots l$, one sees that $\mathbf{H}_n = \mathbb{C}W_n$ (the group algebra of W_n). In this sense, we say that $\mathbf{H}_n$ is a deformation of $\mathbb{C}W_n$.

Remark 2.2.3. We see that when $l = 1$, then $\mathbf{H}_n$ is an Iwahori-Hecke algebra of type A_{n-1}, and when $l = 2$, it is an Iwahori-Hecke algebra of type B_n. This is in accordance with Remark 2.1.2.

Parametrisation of the ordinary irreducible representations

Let K be the field of fractions of A. We set $\mathbf{H}_{K,n} = K \otimes_A \mathbf{H}_n$. We denote by $\mathrm{Irr}(\mathbf{H}_{K,n})$ the set of irreducible $\mathbf{H}_{K,n}$-modules.

Theorem 2.2.4 ([7])**.** *The algebra $\mathbf{H}_{K,n}$ is split semi-simple.*

As a consequence, Tits' deformation theorem implies that the irreducible representations of $\mathbf{H}_{K,n}$ are in one-to-one correspondence with the irreducible representations of the complex reflection group $W_n = G(l, 1, n)$ over $\mathbb{K} = \mathrm{Frac}(R)$. By Section 2.1.3, these representations are in one-to-one correspondence with the set of l-partitions of n. Therefore, we can write

$$\mathrm{Irr}(\mathbf{H}_{K,n}) = \left\{ E^{\boldsymbol{\lambda}} ; \boldsymbol{\lambda} \vdash_l n \right\}. \tag{2.2}$$

The representations $E^{\boldsymbol{\lambda}}$, $\boldsymbol{\lambda} \vdash_l n$, are called the Specht representations. The correspondence $\Pi_l(n) \to \mathrm{Irr}(\mathbf{H}_{K,n}), \boldsymbol{\lambda} \mapsto E^{\boldsymbol{\lambda}}$ is explicitly described in [7, Section 3].

[1] This is needed to prove that Tit's deformation theorem applies, and deduce the parametrisation (2.2).

Specialisations

Definition 2.2.5. Let k be a commutative ring with unity. A *specialisation* is a ring homomorphism $\theta : A \longrightarrow k$.

Remark 2.2.6. The ring k is then turned into an A-module via θ. The *specialised* algebra $\mathbf{H}^{\theta}_{k,n}$ is then defined by $\mathbf{H}^{\theta}_{k,n} = k \otimes_A \mathbf{H}_n$. Denote $\eta = \theta(u)$ and $\eta_i = \theta(V_i)$ for all $i \in [\![, 1, l]\!]$. Then $\mathbf{H}^{\theta}_{k,n}$ is canonically isomorphic to the Ariki-Koike algebra $\mathbf{H}_{k,n}(\eta, \eta_1, \dots, \eta_l)$ defined (over k) as in Definition 2.2.1 but replacing the indeterminates $u, V_1, \dots, V_l$ by $\eta, \eta_1, \dots, \eta_l$ respectively.

In fact, every Ariki-Koike algebra we will study from now on will have parameters in k. Therefore, it will be seen as a particular specialisation of the generic Ariki-Koike algebra $\mathbf{H}_n$.

In their original paper [7], Ariki and Koike also determined a basis for $\mathbf{H}_n$ as an A-module. Bremke and Malle [13] and Malle and Mathas [114] have then studied $\mathbf{H}_n$ as a symmetric algebra. Moreover, it has a cellular structure, as is explicited in the original paper by Graham and Lehrer [63, Section 5], see also [34]. In [117], Mathas gives a thorough review of the representation theory of Ariki-Koike algebras.

2.2.2 Semisimplicity criterion and modular representation theory

Characterisation of the non-semisimple specialisations

If we want to study modular representation theory of Ariki-Koike algebras, we first need to know precisely which values of the parameters yield non-semisimple algebras. It turns out that we have an explicit criterion to check whether such an algebra is semisimple or not. The result is due to Ariki, and is taken from [3].

Theorem 2.2.7. *Let $\theta : A \longrightarrow k$ be a specialisation, with $k = \mathrm{Frac}(\theta(A))$. Denote $\eta = \theta(u) \neq 0$ and $\eta_i = \theta(V_i)$, for all $i \in [\![1, l]\!]$. Then the specialised algebra $\mathbf{H}_{k,n}(\eta, \eta_1, \dots, \eta_l) = k \otimes_A \mathbf{H}_n$ is (split) semi-simple if and only if*

$$\Big(\prod_{-n<d<n} \prod_{1 \leq i < j \leq l} (\eta^d \eta_i - \eta_j)\Big)\Big(\prod_{1 \leq i \leq n} (1 + \eta + \dots + \eta^{i-1})\Big) \neq 0.$$

Remark 2.2.8. Roughly speaking, the second factor checks if η is a non-trivial root of unity, and the first factor establishes a relation between the parameters η_i and the parameter η.

As a matter of fact, we have the following reduction result by Dipper and Mathas [35]:

Theorem 2.2.9. *Let η and η_i $(i = 1, \dots, l)$ be as in Theorem 2.2.7. Denote $\mathcal{E}$ the multiset $\{\eta_1, \dots, \eta_l\}$ and suppose that there exists a partition $\mathcal{E} = \mathcal{E}_1 \sqcup \dots \sqcup \mathcal{E}_s$ such that*

$$\prod_{1 \leq \alpha < \beta \leq s} \prod_{(\eta_i, \eta_j) \in \mathcal{E}_\alpha \times \mathcal{E}_\beta} \prod_{-n<N<n} (\eta^N \eta_i - \eta_j) \neq 0.$$

Then $\mathbf{H}_{k,n}(\eta,\eta_1,\ldots,\eta_l)$ *is Morita equivalent to the algebra*

$$\bigoplus_{\substack{n_1+\cdots+n_s=n\\ n_1,\ldots,n_s\geq 0}} \mathbf{H}_{k,n_1}(\eta,\mathcal{E}_1)\otimes_k\cdots\otimes_k\mathbf{H}_{k,n_s}(\eta,\mathcal{E}_s).$$

As a consequence, in order to study non semi-simple Ariki-Koike algebras, it is sufficient to consider the specialisations $\mathbf{H}_{k,n}(\eta,\eta_1,\ldots,\eta_l)$ of $\mathbf{H}_n$ defined via

$$\begin{array}{rccl} \theta: & A & \longrightarrow & k\\ & u & \longmapsto & \eta\\ & V_i & \longmapsto & \eta^{s_i}, \end{array}$$

where η has multiplicative order e, and $s_i\in\mathbb{Z}$ for all $i\in[\![1,l]\!]$. In the case where η is not a root of unity, we set $e=\infty$. In fact, each specialisation considered from now on will be characterised by a pair $(e,\mathbf{s})$ where $e\in\mathbb{Z}_{>1}\sqcup\{\infty\}$ and $\mathbf{s}=(s_1,\ldots,s_l)\in\mathbb{Z}^l$. We will denote equally $\mathbf{H}_{k,n}(\eta,\eta^{s_1},\ldots,\eta^{s_j})=\mathbf{H}_{k,n}^{(e,\mathbf{s})}$ the specialised Ariki-Koike algebra corresponding to $(e,\mathbf{s})$. We also denote $\theta_{(e,\mathbf{s})}$ the associated specialisation map.

The decomposition matrix

Consider the specialisation $\theta_{(e,\mathbf{s})}: A\longrightarrow k$ with $k=\mathrm{Frac}(\theta_{(e,\mathbf{s})}(A))$. In accordance with [1], [2] (or [54] for $l=1,2$), there is an associated decomposition map $d_{\theta_{(e,\mathbf{s})}}: R_0(\mathbf{H}_{K,n})\longrightarrow R_0(\mathbf{H}_{k,n}^{(e,\mathbf{s})})$, and we can write

$$d_{\theta_{(e,\mathbf{s})}}([E^{\boldsymbol{\lambda}}])=\sum_{M\in\mathrm{Irr}(\mathbf{H}_{k,n}^{(e,\mathbf{s})})} d_{\boldsymbol{\lambda},M}[M].$$

Definition 2.2.10. The *decomposition matrix* of $\mathbf{H}_{k,n}^{(e,\mathbf{s})}$ is the matrix

$$D_{(e,\mathbf{s})}=(d_{\boldsymbol{\lambda},M})_{\substack{\boldsymbol{\lambda}\vdash_l n\\ M\in\mathrm{Irr}(\mathbf{H}_{k,n}^{(e,\mathbf{s})})}}.$$

The elements $d_{\boldsymbol{\lambda},M}$ are called the *decomposition numbers* of $\mathbf{H}_{k,n}^{(e,\mathbf{s})}$. If the specialised algebra is semi-simple, the decomposition map is trivial and $D_{(e,\mathbf{s})}$ is the identity matrix. In general, this matrix has a rectangular shape, since $|\mathrm{Irr}(\mathbf{H}_{k,n}^{(e,\mathbf{s})})|\leq|\left\{E^{\boldsymbol{\lambda}};\boldsymbol{\lambda}\vdash_l n\right\}|$ ([1], [2]). For simplicity, we say that $\boldsymbol{\lambda}$ *appears* in the column C indexed by M if $d_{\boldsymbol{\lambda},M}\neq 0$.

In the sequel, we will be interested with the shape of this decomposition matrix. When there is no possible confusion, we will drop the superscript $(e,\mathbf{s})$ in $\mathbf{H}_{k,n}^{(e,\mathbf{s})}$. One of the classic problems is to find an indexation of the simple $\mathbf{H}_{k,n}$-modules so that D is upper unitriangular, that is,

$$D = \begin{array}{cc} \overbrace{\hphantom{1\quad 0\quad \dots\quad 0}}^{\mathrm{Irr}(\mathbf{H}_{k,n})} & \\ \begin{pmatrix} 1 & 0 & \dots & 0 \\ \star & 1 & \dots & 0 \\ \vdots & \star & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 \\ \vdots & \vdots & & \star \\ \vdots & \vdots & \dots & \vdots \end{pmatrix} & \left.\vphantom{\begin{matrix}1\\1\\1\\1\\1\\1\end{matrix}}\right\}\Pi_l(n) \end{array}$$

More precisely, we ask for an order $\leq$ on $\mathrm{Irr}(\mathbf{H}_{K,n})$, or equivalently on $\Pi_l(n)$ by (2.2), such that D has the above shape with respect to this order, that is, $i > j \Rightarrow \boldsymbol{\lambda}_i \leq \boldsymbol{\lambda}_j$, if $\boldsymbol{\lambda}_i \vdash_l n$ parametrises the i-th row of D; and the extra condition that $d_{\boldsymbol{\lambda},M} = 1$ if the index of the column labeled by M is the index of the row labeled by $\boldsymbol{\lambda}$. These problems have been solved in [53], using the theory of canonical basic sets. This approach enables us find a bijection between $\mathrm{Irr}(\mathbf{H}_{k,n})$ and a subset of $\Pi_l(n)$, and therefore to label the simple $\mathbf{H}_{k,n}$-modules by certain l-partitions, the *Uglov multipartitions*.

2.2.3 Canonical basic sets

Order induced by the a-function

As previously mentioned in Section 2.2.1, $\mathbf{H}_n$ has the structure of a symmetric algebra. This means the existence of symmetrising form, which permits to associate to each simple $\mathbf{H}_{K,n}$-module $E^{\boldsymbol{\lambda}}$ its Schur element $\mathbf{c}^{\boldsymbol{\lambda}}$. Explicit formulas for computing $\mathbf{c}^{\boldsymbol{\lambda}}$ have been given independently in [52] and [116]. According to [28], $\mathbf{c}^{\boldsymbol{\lambda}}$ is an element of $\mathbb{Z}[u^{\pm 1}, V_1^{\pm 1}, \dots, V_l^{\pm 1}]$ (that is, a Laurent polynomial in the variables $u, V_1, \dots, V_l$, see also [54, Proposition 7.3.9]).

Now fix $\mathbf{m} = (m_1, \dots, m_l) \in \mathbb{Q}^l$. One can define the degree of $\mathbf{c}^{\boldsymbol{\lambda}}$ by setting

$$\deg_{\mathbf{m}}(u^p V_1^{p_1} \dots V_l^{p_l}) = p + m_1 p_1 + \dots + m_l p_l, \quad \text{and}$$

$$\deg_{\mathbf{m}}(\mathbf{c}^{\boldsymbol{\lambda}}) = \min\left\{ \deg_{\mathbf{m}}(u^p V_1^{p_1} \dots V_l^{p_l}) \;;\; u^p V_1^{p_1} \dots V_l^{p_l} \text{ is a monomial appearing in } \mathbf{c}^{\boldsymbol{\lambda}} \right\}.$$

Such an element $\mathbf{m}$ is then called a *weight sequence*. Note that this definition of the degree is different from the usual one for Laurent polynomials (namely, one usually takes the maximum of the degrees of the monomials).

Extending Lusztig's [110] definition of the **a**-function, one can then introduce *generalised* **a**-*invariants* for the modules $E^{\boldsymbol{\lambda}}$. We simply set $\mathbf{a}^{\mathbf{m}}(\boldsymbol{\lambda}) = -\deg_{\mathbf{m}}(\mathbf{c}^{\boldsymbol{\lambda}})$. The map $E^{\boldsymbol{\lambda}} \mapsto \mathbf{a}^{\mathbf{m}}(\boldsymbol{\lambda})$ is then called a *generalised* **a**-*function*, and coincides with Lusztig's **a**-function when $l = 1$ and $\mathbf{m} = m_1 = 1$.

Remark 2.2.11. The weight sequence $\mathbf{m}$ we just introduced also has another algebraic meaning. In [16], Broué and Malle have introduced the notion of "cyclotomic" Hecke algebra. In the case of an Ariki-Koike algebra, see [53, Chapter 5], this is a one-parameter specialisation of $\mathbf{H}_n$, parametrised by a pair $(\mathbf{m}, t) \in \mathbb{Q}^l \times \mathbb{Q}$. Thanks to Theorem 2.2.7, any cyclotomic specialisation is known to be semi-simple. Note that in [53], the generalised $\mathbf{a}$-function is defined on this cyclotomic specialisation. Interestingly, when $e < \infty$, any non semi-simple algebra $\mathbf{H}_{k,n}^{(e,\mathbf{s})} = \mathbf{H}_{k,n}$ can be obtained by specialising a certain cyclotomic algebra. In fact, if $m_i = s_i - e(i-1)/pl$ with $\gcd(p, e) = 1$ and t is such that $tm_i \in \mathbb{Z}$ for all i, we have a cyclotomic algebra $\mathbf{H}_{\mathbb{K}(y),n}$ depending on an indeterminate y, which can be specialised to $\mathbf{H}_{k,n}$ via $y \mapsto \xi^{1/t} := \exp(2ip\pi/et)$. In other terms, the following diagram commutes:

$$\begin{tikzcd}
\mathbf{H}_n \arrow[dd, "\theta_{(e,\mathbf{s})}"'] \arrow[dr, "\theta_y"] & \\
 & \mathbf{H}_{\mathbb{K}(y),n} \arrow[dl, "\tilde{\theta}"] \\
\mathbf{H}_{k,n} &
\end{tikzcd}$$

$$\begin{array}{llcll}
\text{where} \quad \theta_y : & A & \longrightarrow & \mathbb{K}(y) & \\
 & u & \longmapsto & y^t & \text{with } t \text{ such that } t(s_i - \frac{e(i-1)}{pl}) \in \mathbb{Z}, \\
 & V_i & \longmapsto & y^{t(s_i - \frac{e(i-1)}{pl})}\zeta^{i-1} & \text{for } i \in [\![1, l]\!]
\end{array}$$

is the cyclotomic specialisation, where $\zeta = \exp(2\pi\sqrt{-1}/l)$,

$$\begin{array}{llcll}
\text{and} \quad \tilde{\theta} : & \mathbb{K}(y) & \longrightarrow & k & \text{such that } \xi^{1/t} \in k \\
 & y & \longmapsto & \xi^{1/t}. &
\end{array}$$

Now, the $\mathbf{a}$-invariants induce an order on Specht modules, namely $E^{\boldsymbol{\lambda}} \sqsubseteq E^{\boldsymbol{\mu}} \Leftrightarrow [\boldsymbol{\lambda} = \boldsymbol{\mu}$ or $\mathbf{a}^{\mathbf{m}}(\boldsymbol{\lambda}) < \mathbf{a}^{\mathbf{m}}(\boldsymbol{\mu})]$. The general notion of canonical basic sets requires an order on the Specht modules. In the case of Ariki-Koike algebras, it is natural to use this algebraic order. In fact, we will use the combinatorial order $\ll_{\mathbf{m}}$, defined in Section 1.2, which contains the order $\sqsubseteq$ above.

Recall that we have set $n_{\mathbf{m}}(\boldsymbol{\lambda}) = \sum_{1 \leq i \leq T(\boldsymbol{\lambda}, \mathbf{m})} (i-1)\mathfrak{b}_{\mathbf{m}}^i(\boldsymbol{\lambda})$ in Section 1.2.2, Formula 1.1. By [53, Proposition 5.5.11], we can compute the $\mathbf{a}$-invariant of $\boldsymbol{\lambda}$ using symbols, namely $\mathbf{a}^{\mathbf{m}}(\boldsymbol{\lambda}) = t(n_{\mathbf{m}}(\boldsymbol{\lambda}) - n_{\mathbf{m}}(\boldsymbol{\emptyset}))$. As a direct consequence, we have the following compatibility property, see [53, Proposition 5.7.7] for the proof:

$$[\boldsymbol{\lambda} \ll_{\mathbf{m}} \boldsymbol{\mu} \text{ and } \boldsymbol{\lambda} \neq \boldsymbol{\mu}] \quad \Rightarrow \quad \mathbf{a}^{\mathbf{m}}(\boldsymbol{\lambda}) < \mathbf{a}^{\mathbf{m}}(\boldsymbol{\mu}). \tag{2.3}$$

Remark 2.2.12. The order $\ll_{\mathbf{m}}$ on symbols has the advantage of being easier to handle, since it is purely combinatorial. Besides, it naturally appears in the representation theory of the complex reflection groups of type $G(l, 1, n)$.

For instance, when $l = 2$, Geck and Iancu showed in [51, Theorem 7.11] that this order is compatible with an order $\preceq_L$, defined (in [47]) on $\mathrm{Irr}(G(l, 1, n))$ using Lusztig's families (and related to the order $\leq_{\mathcal{LR}}$ defining Kazhdan-Lusztig cells). In fact, they showed that in general, one has $\boldsymbol{\lambda} \preceq_L \boldsymbol{\mu} \Rightarrow \boldsymbol{\lambda} \ll_{\mathbf{m}} \boldsymbol{\mu}$, and that in some particular cases, both orders are equivalent. Note that the version of the order $\ll_{\mathbf{m}}$ defined in [51] is slightly different from the one we have defined in Chapter 1.

Moreover, Chlouveraki, Gordon and Griffeth have used the compatibility property (2.3) above to deduce information on the decomposition of standard modules of Cherednik algebras [2], [26, Theorem 5.7]. Also, Liboz showed in [105] that the order $\ll_{\mathbf{m}}$ contains the order induced by the $\mathbf{c}$-*function* on $\mathrm{Irr}(G(l,1,n))$ used in the representation theory of Cherednik algebras.

Canonical basic sets

We can now state the definition of a canonical basic set in the sense of [46], for both the order $\ll_{\mathbf{m}}$ and the one induced by the $\mathbf{a}$-invariants. Fix $\mathbf{s} \in \mathbb{Z}^l$, $e \in \mathbb{Z}_{>1}$, and a weight sequence $\mathbf{m} \in \mathbb{Q}^l$. Consider the specialised algebra $\mathbf{H}_{k,n}^{(e,\mathbf{s})} = \mathbf{H}_{k,n}$. For $M \in \mathrm{Irr}(\mathbf{H}_{k,n})$, set $\mathscr{S}(M) = \{\boldsymbol{\lambda} \vdash_l n \,|\, d_{\boldsymbol{\lambda},M} \neq 0\}$.

Definition 2.2.13. Assume that the following conditions hold:

1. For $M \in \mathrm{Irr}(\mathbf{H}_{k,n})$, there exists a unique element $\boldsymbol{\lambda}_M \in \mathscr{S}(M)$ such that for all $\boldsymbol{\mu} \in \mathscr{S}(M)$, $\boldsymbol{\lambda}_M \ll_{\mathbf{m}} \boldsymbol{\mu}$ (resp. $\mathbf{a}^{\mathbf{m}}(\boldsymbol{\lambda}_M) < \mathbf{a}^{\mathbf{m}}(\boldsymbol{\mu})$ or $\boldsymbol{\lambda} = \boldsymbol{\mu}$).
2. The map $\mathrm{Irr}(\mathbf{H}_{k,n}) \to \Pi_l(n)$, $M \mapsto \boldsymbol{\lambda}_M$ is injective.
3. We have $d_{\boldsymbol{\lambda}_M,M} = 1$, for all $M \in \mathrm{Irr}(\mathbf{H}_{k,n})$.

Then the set $\mathscr{B} := \{\boldsymbol{\lambda}_M; M \in \mathrm{Irr}(\mathbf{H}_{k,n})\} \subseteq \Pi_l(n)$ is in one-to-one correspondence with $\mathrm{Irr}(\mathbf{H}_{k,n})$. It is called a *canonical basic set* for $(\mathbf{H}_{k,n}, \mathbf{s})$ with respect to $\ll_{\mathbf{m}}$ (resp. with respect to the $\mathbf{a}$-function).

Remark 2.2.14. As a direct consequence, if there exists a canonical basic set for $(\mathbf{H}_{k,n}, \mathbf{s})$ with respect to $\ll_{\mathbf{m}}$ (or with respect to the $\mathbf{a}$-function), it is unique. Moreover, the three conditions of Definition 2.2.13 encode the fact that D is upper unitriangular with respect to $\ll_{\mathbf{m}}$.

The question of determining canonical basic sets for $(\mathbf{H}_{k,n}, \mathbf{s})$ has been solved in some cases. First, in [75], Jacon has studied the case where $m_i = s_i - e(i-1)/l$ (which is also when $\theta_{(e,\mathbf{s})}$ can be decomposed in a cyclotomic specialisation and a non semi-simple specialisation, as noticed in Remark 2.2.11), and in [53], Geck and Jacon have explained the more general case where $m_i = s_i - v_i$ with some restrictions on $(v_1, \dots, v_l)$. In fact, in Chapter 4, we recall these results (Theorems 4.2.1 and 4.2.3), and we generalise them.

2.3 Rational Cherednik algebras

We now focus on another algebraic structure associated to the complex reflection group W_n, namely the rational Cherednik algebra, sometimes also called double affine

[2] Cherednik algebras will be introduced in upcoming Section 2.3.

Hecke algebra. It has been introduced by Etingof and Ginzburg in [37] as a rational degeneration of the original object defined by Cherednik in [25]. Note that it exists for any complex reflection group. In the particular case of W_n, it is called cyclotomic rational Cherednik algebra.

2.3.1 Definition

Fix $l \in \mathbb{Z}_{>0}$, $e \in \mathbb{Z}_{>1}$ and $s = (s_1, \dots, s_l) \in \mathbb{Z}^l$. Recall the notations of Section 2.1.2, and the decomposition of $\mathcal{S}_n$ (2.1). For $1 \leq p \leq l-1$, define the map $c : \mathcal{S}_n \longrightarrow \mathbb{C}$ by

$$\begin{aligned} c(\varepsilon_k^p) &= \sum_{p'=1}^{l-1} (\zeta^{-pp'} - 1) \left(\frac{s_{p+1} - s_p}{e} - \frac{1}{l} \right) \quad \text{and} \\ c(t_{ij}\varepsilon_i\varepsilon_j^{-1}) &= -\frac{1}{e}. \end{aligned} \tag{2.4}$$

Let $T(V \oplus V^*)$ be the tensor algebra of $V \oplus V^*$.

Definition 2.3.1. The *cyclotomic rational Cherednik algebra* $H_{c,n}$ is the quotient of the smash product $T(V \oplus V^*) \rtimes W_n$ by the relations:

- $[x, x'] = 0$,
- $[y, y'] = 0$, and
- $[y, x] = \langle x, y \rangle - \sum_{s \in \mathcal{S}_n} c(s) \dfrac{\langle \alpha_s, y \rangle \langle x, v_s \rangle}{\langle \alpha_s, v_s \rangle} s$,

for $x, x' \in V^*$ and $y, y' \in V$.

The algebra $H_{c,n}$ contains the three subalgebras $\mathbb{C}W_n$, $\mathbb{C}[V]$ and $\mathbb{C}[V^*]$. We have the following Poincaré-Birkhoff-Witt theorem for rational Cherednik algebras.

Theorem 2.3.2 ([37])**.** *The multiplication yields an isomorphism of vector spaces*

$$\mathbb{C}[V] \otimes_{\mathbb{C}} \mathbb{C}W_n \otimes_{\mathbb{C}} \mathbb{C}[V^*] \xrightarrow{\sim} H_{c,n}.$$

2.3.2 The category $\mathcal{O}$

A nice category of $H_{c,n}$-modules has been introduced by Ginzburg, Guay, Opdam and Rouquier in [57]. For the rest of the notions on rational Cherednik algebras, one can also refer to [124].

Definition 2.3.3. The category $\mathcal{O}_{c,n}$ is the full subcategory of $H_{c,n}$-modules that are finitely generated as $\mathbb{C}[V]$-modules and V-locally nilpotent. We also set

$$\mathcal{O}_c = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \mathcal{O}_{c,n}.$$

Consider now the complexified Grothendieck group $[\mathcal{O}_{c,n}]$ of $\mathcal{O}_{c,n}$. It is a $\mathbb{C}$-vector space. We also denote $[\mathcal{O}_c]$ the complexified Grothendieck group of $\mathcal{O}_c$.

Recall that we have denoted $\mathrm{Irr}_{\mathbb{C}}(W_n) = \{\mathscr{E}^{\boldsymbol{\lambda}} \; ; \; \boldsymbol{\lambda} \vdash_l n\}$. For each $\boldsymbol{\lambda} \vdash_l n$, we can view $\mathscr{E}^{\boldsymbol{\lambda}}$ as a $\mathbb{C}W_n \ltimes \mathbb{C}[V^*]$-module. We then define the *standard* module $\Delta(\boldsymbol{\lambda})$ as the following induced $H_{c,n}$-module:

$$\Delta(\boldsymbol{\lambda}) = H_{c,n} \otimes_{\mathbb{C}W_n \ltimes \mathbb{C}[V^*]} \mathscr{E}^{\boldsymbol{\lambda}}.$$

These standard modules are indecomposable in $\mathcal{O}_{c,n}$. Moreover, let $L(\boldsymbol{\lambda})$ be the head of $\Delta(\boldsymbol{\lambda})$, that is, the largest semisimple quotient of $\Delta(\boldsymbol{\lambda})$. It turns out that $L(\boldsymbol{\lambda})$ is simple. Moreover, the set of all $L(\boldsymbol{\lambda})$, for $\boldsymbol{\lambda} \vdash_l n$, gives a complete set of non-isomorphic simple objects in $\mathcal{O}_{c,n}$.

According to [57], $\mathcal{O}_{c,n}$ is a *highest weight category* with respect to the standard objects $\Delta(\boldsymbol{\lambda})$ and the order induced by the $\mathbf{c}$-*function* (see also [105] or [124] for the definitions). This implies in particular that

$$\{[\Delta(\boldsymbol{\lambda})] \; ; \; \boldsymbol{\lambda} \vdash_l n\} \quad \text{and} \quad \{[L(\boldsymbol{\lambda})] \; ; \; \boldsymbol{\lambda} \vdash_l n\}$$

are two bases of the vector space $[\mathcal{O}_{c,n}]$.

2.3.3 i-induction and i-restriction

In their paper [10], Bezrukavnikov and Etingof have defined a pair of biadjoint functors for rational Cherednik algebras. Their construction being quite technical, we do not recall it precisely.

In the case of W_n, this is made using the natural embedding of W_{n-1} into W_n, which identifies W_{n-1} with the parabolic subgroup of W_n given by the stabilizer of the point $(0, \ldots, 0, 1) \in \mathbb{C}^n$. Following [10, Section 3.5], this gives rise to functors

$$F(n) : \mathcal{O}_{c,n-1} \longrightarrow \mathcal{O}_{c,n} \quad \text{(called parabolic induction)}$$

and

$$E(n) : \mathcal{O}_{c,n} \longrightarrow \mathcal{O}_{c,n-1} \quad \text{(called parabolic restriction).}$$

These definitions have been refined by Shan in [124, Definition 4.2] to define i-induction and i-restriction functors, for $i \in [\![0, e-1]\!]$. In fact, for all $a(z) \in \mathbb{C}(z)$, she first introduces a particular exact functor

$$Q_{n,a(z)} : \mathcal{O}_{c,n} \longrightarrow \mathcal{O}_{c,n},$$

and then gives the following definition.

Definition 2.3.4. Let $i \in [\![0, e-1]\!]$. Then the *i-induction* functor $F_i(n) : \mathcal{O}_{c,n-1} \longrightarrow \mathcal{O}_{c,n}$ and the *i-restriction* functor $E_i(n) : \mathcal{O}_{c,n} \longrightarrow \mathcal{O}_{c,n-1}$ are defined by:

$$F_i(n) = \bigoplus_{a(z) \in \mathbb{C}(z)} Q_{n-1,a(z)/(z-\xi^i)} \circ F(n) \circ Q_{n,a(z)}$$

and

$$E_i(n) = \bigoplus_{a(z)\in\mathbb{C}(z)} Q_{n,a(z)/(z-\xi^i)} \circ E(n) \circ Q_{n-1,a(z)},$$

where $\xi = \exp(2\pi\sqrt{-1}/e)$.

They are biadjoint functors. Moreover, we have the following properties [124, Proposition 4.4].

Proposition 2.3.5.

1. *For all* $\boldsymbol{\lambda} \vdash_l n$*, we have*

$$F_i(n)([\Delta(\boldsymbol{\lambda})]) = \sum_{\mathrm{res}(\boldsymbol{\mu}\backslash\boldsymbol{\lambda})=i} [\Delta(\boldsymbol{\mu})] \quad \textit{and} \quad E_i(n)([\Delta(\boldsymbol{\lambda})]) = \sum_{\mathrm{res}(\boldsymbol{\lambda}\backslash\boldsymbol{\mu})=i} [\Delta(\boldsymbol{\mu})].$$

2. *We have*

$$F(n) = \bigoplus_{0\leq i\leq e-1} F_i(n) \quad \textit{and} \quad E(n) = \bigoplus_{0\leq i\leq e-1} E_i(n).$$

We will see in the next chapter that the action of these i-induction and i-restriction operators on the simple modules $L(\boldsymbol{\lambda})$ define a certain *crystal* structure on $[\mathcal{O}_c]$, which can be made explicit.

Chapter 3

Fock spaces representations of $\mathcal{U}_q(\widehat{\mathfrak{sl}_e})$

This chapter is a review of the theory of crystals and canonical bases for Fock spaces. First of all, we give some background on the representation theory of the quantum algebra $\mathcal{U}_q(\widehat{\mathfrak{sl}_e})$. We also introduce a particular combinatorial representation: the Fock space $\mathcal{F}_{\mathbf{s}}$. In his theory of crystals [86], Kashiwara introduced a combinatorial structure on representation of quantum algebras. This gives in particular the existence of a crystal basis, crystal graph, and a canonical basis for $\mathcal{F}_{\mathbf{s}}$. We also explain how this theory is valid in the limit case $e \longrightarrow \infty$, using the algebra $\mathcal{U}_q(\mathfrak{sl}_\infty)$.

Then, we state in Section 3.3.1 maybe the most significant result we use in this thesis, namely Ariki's theorem (Theorem 3.3.1). It explains how the representations $\mathcal{F}_{\mathbf{s}}$ of $\mathcal{U}_q(\widehat{\mathfrak{sl}_e})$ controls the modular representations of Ariki-Koike algebras, enabling the computation of their decomposition matrices. We also recall a result proved by Shan and refined by Losev [107], which gives an interpretation of the crystal of $\mathcal{F}_{\mathbf{s}}$ in terms of rational Cherednik algebras.

3.1 The quantum affine algebra $\mathcal{U}_q(\widehat{\mathfrak{sl}_e})$

In this section, we recall the definition of $\mathcal{U}_q(\widehat{\mathfrak{sl}_e})$ as a deformation of $\widehat{\mathfrak{sl}_e}$. We then give some elements on its representation theory, and we explain Kashiwara's crystal and global theory. Our main reference here is the thesis of Yvonne [131]. For a review on Kac-Moody algebras and their quantum deformations in general, on can refer to the book of Hong and Kang [73].

3.1.1 The Kac-Moody algebra $\widehat{\mathfrak{sl}_e}$

Let $e \in \mathbb{Z}_{>1}$, and $\mathfrak{h}$ be an e-dimensional $\mathbb{Q}$-vector space with basis $\{h_0, \dots, h_{e-1}, D\}$. Let $\{\Lambda_0, \dots, \Lambda_{e-1}, \delta\}$ be the basis of $\mathfrak{h}^*$ defined via the pairing $\langle ., . \rangle : \mathfrak{h}^* \times \mathfrak{h} \longrightarrow \mathbb{Q}$ by

$$\langle \Lambda_i, h_j\rangle = \delta_{i,j}, \quad \langle \Lambda_i, D\rangle = \langle \delta, h_i\rangle = 0 \quad \text{and} \quad \langle \delta, D\rangle = 1,$$

for all $0 \leq i, j \leq e-1$.

The elements Λ_i are called the *fundamental weights*. Out of simplicity, we will allow ourselves to index them by $\mathbb{Z}$ by setting $\Lambda_k = \Lambda_{k \mod e}$. We then define the *simple roots* to be the following elements α_i of $\mathfrak{h}^*$:

$$\alpha_i = -\Lambda_{i-1} + 2\Lambda_i - \Lambda_{i+1} + \delta_{i,0}\delta, \tag{3.1}$$

for all $i \in [\![0, e-1]\!]$. Also, wet set

$$a_{ij} = \langle \alpha_i, h_j\rangle, \tag{3.2}$$

that is, the coefficient of Λ_j in α_i.

The set

$$P = \bigoplus_{i=0}^{e-1} \mathbb{Z}\Lambda_i \oplus \mathbb{Z}\delta \quad (\text{resp. } Q = \bigoplus_{i=0}^{e-1} \mathbb{Z}\alpha_i)$$

is called the *weight lattice* (resp. the *root lattice*). Moreover, we define the set of *dominant weights* to be

$$P^+ = \bigoplus_{i=0}^{e-1} \mathbb{Z}_{\geq 0}\Lambda_i \oplus \mathbb{Z}\delta.$$

There is a scalar product $(.|.)$ on $\mathfrak{h}^*$ defined by

$$(\alpha_i|\alpha_j) = a_{ij}, \quad (\Lambda_0|\alpha_i) = \delta_{i,0}, \quad (\Lambda_0|\Lambda_0) = 0, \tag{3.3}$$

for $0 \leq i < j \leq e-1$.

Definition 3.1.1. The Kac-Moody algebra $\widehat{\mathfrak{sl}_e}$ is the $\mathbb{Q}$-algebra defined by

- generators E_i, F_i, G_i, d for $i \in [\![0, e-1]\!]$
- relations:
  - $[G_i, G_j] = 0$,
  - $[G_i, E_j] = a_{ij}E_j$,
  - $[G_i, F_j] = -a_{ij}F_j$,
  - $[E_i, F_j] = \delta_{i,j}G_i$,
  - $[d, G_i] = 0$,
  - $[d, E_i] = \delta_{i,0}E_i$,
  - $[d, F_i] = -\delta_{i,0}F_i$,
  - $(\text{ad } E_i)^{1-a_{ij}}(E_j) = (\text{ad } F_i)^{1-a_{ij}}(F_j) = 0$ (the *Serre relations*).

  for all $i, j \in [\![0, e-1]\!]$.

3.1.2 The quantum algebra $\mathcal{U}_q(\widehat{\mathfrak{sl}_e})$

Let q be an indeterminate.

q-integers

We will need the following notations. For $k \in \mathbb{Z}_{\geq 0}$, we denote $[k] = \dfrac{q^k - q^{-k}}{q - q^{-1}}$ and $[k]! = [k][k-1]\dots[1]$. $[k]$ is called a *q-integer*. Then, for $1 \leq k \leq m$, we set

$$\begin{bmatrix} m \\ k \end{bmatrix} = \frac{[m]!}{[m-k]![k]!},$$

the q-analogues of the binomial coefficients.

Moreover, we associate to each element x of a unital $\mathbb{Q}(q)$-algebra the elements

$$\{x\} = \frac{x - x^{-1}}{q - q^{-1}}, \quad \left\{ \begin{matrix} x \\ 0 \end{matrix} \right\} = 1 \quad \text{and} \quad \left\{ \begin{matrix} x \\ k \end{matrix} \right\} = \frac{\{x\}\{q^{-1}x\}\dots\{q^{-(k-1)}x\}}{[k]!} \text{ for } k \in \mathbb{Z}_{>0}.$$

The quantum affine algebra of type A

Definition 3.1.2. The quantum affine algebra $\mathcal{U}_q(\widehat{\mathfrak{sl}_e})$ is the $\mathbb{Q}(q)$-algebra defined by

- generators $e_i, f_i, t_i, t_i^{-1}, \mathfrak{d}, \mathfrak{d}^{-1}$ for $i \in [\![0, e-1]\!]$
- relations:
  - $t_i t_i^{-1} = t_i^{-1} t_i = 1$,
  - $\mathfrak{d}\mathfrak{d}^{-1} = \mathfrak{d}^{-1}\mathfrak{d} = 1$,
  - $t_i e_j t_i^{-1} = q^{a_{ij}} e_j$,
  - $t_i f_j t_i^{-1} = q^{-a_{ij}} f_j$,
  - $[e_i, f_j] = \delta_{i,j} \dfrac{t_i - t_i^{-1}}{q - q^{-1}}$,
  - $[\mathfrak{d}, e_i] = \delta_{i,0} e_i$,
  - $[\mathfrak{d}, f_i] = \delta_{i,0} f_i$,
  - $[\mathfrak{d}, t_i] = 0$,
  - $\displaystyle\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix} e_i^{1-a_{ij}-k} e_j e_i^k = \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix} f_i^{1-a_{ij}-k} f_j f_i^k = 0$ (the *q-Serre relations*).

  for all $i, j \in [\![0, e-1]\!]$.

Moreover, we denote by $\mathcal{U}_q'(\widehat{\mathfrak{sl}_e})$ the parabolic subalgebra of $\mathcal{U}_q(\widehat{\mathfrak{sl}_e})$ generated by the elements e_i, f_i, t_i, t_i^{-1}.

The elements

$$e_i^{(k)} = \frac{e_i^k}{[k]!} \quad \text{and} \quad f_i^{(k)} = \frac{f_i^k}{[k]!}$$

are called the *divided powers* of e_i and f_i.

Remark 3.1.3. One sees that the relations, which make the parameter q appear, are similar to the ones defining $\widehat{\mathfrak{sl}_e}$. In fact, one can view $\mathcal{U}_q(\widehat{\mathfrak{sl}_e})$ as a deformation of the universal enveloping algebra $\mathcal{U}(\widehat{\mathfrak{sl}_e})$ of $\widehat{\mathfrak{sl}_e}$: we recover $\mathcal{U}(\widehat{\mathfrak{sl}_e})$ by taking $q = 1$.

A first structural result is the following Poincaré-Birkhoff-Witt theorem. Denote $\mathcal{U}_q(\widehat{\mathfrak{sl}_e})^+$ (resp. $\mathcal{U}_q(\widehat{\mathfrak{sl}_e})^-$, resp. $\mathcal{U}_q(\mathfrak{h})$) the parabolic subalgebra of $\mathcal{U}_q'(\widehat{\mathfrak{sl}_e})$ generated by the elements e_i (resp. f_i, resp. $t_i^{\pm 1}$) for all $i \in [\![0, e-1]\!]$.

Theorem 3.1.4. *The multiplication yields an isomorphism*

$$\mathcal{U}_q(\widehat{\mathfrak{sl}_e})^+ \otimes \mathcal{U}_q(\mathfrak{h}) \otimes \mathcal{U}_q(\widehat{\mathfrak{sl}_e})^- \xrightarrow{\sim} \mathcal{U}_q'(\widehat{\mathfrak{sl}_e}).$$

Moreover, we have the following additional tensor structure on $\mathcal{U}_q(\widehat{\mathfrak{sl}_e})$.

Proposition 3.1.5. *There is a coproduct Δ, given by the following formulas, which turns $\mathcal{U}_q(\widehat{\mathfrak{sl}_e})$ into a Hopf algebra:*

- $\Delta(e_i) = e_i \otimes t_i + 1 \otimes e_i$,
- $\Delta(f_i) = f_i \otimes 1 + t_i^{-1} \otimes f_i$,
- $\Delta(t_i) = t_i \otimes t_i$,
- $\Delta(t_i^{-1}) = t_i^{-1} \otimes t_i^{-1}$,
- $\Delta(\mathfrak{d}) = \mathfrak{d} \otimes 1 + 1 \otimes \mathfrak{d}$.

3.1.3 Integrable representations and highest weight theory

Weight spaces, weight vectors and cyclic modules

Let M be a $\mathcal{U}_q(\widehat{\mathfrak{sl}_e})$-module, and $\Lambda \in P$ be a weight. Write $\Lambda = \sum_{i=0}^{e-1} a_i \Lambda_i + d\delta$, with $a_0, \dots, a_{e-1}, d \in \mathbb{Z}$. Denote by M_Λ the subspace of M defined by

$$M_\Lambda = \{m \in M \mid \mathfrak{d}.m = dm \quad \text{and} \quad t_i.m = q^{a_i} m \quad \text{for all } i \in [\![0, e-1]\!]\}$$

is called a Λ-*weight space*. If M_Λ is non-trivial, then Λ is called a *weight for* M, and the non-zero elements of M_Λ are called Λ-*weight vectors*. If $u \in M_\Lambda$, we write $\mathrm{wt}(u) = \Lambda$, the *weight* of u.

A *highest weight vector* of weight Λ is an element u of M_Λ verifying $e_i.u = 0$ for all $i \in [\![0, e-1]\!]$. Then we say that M is a *cyclic module of highest weight* Λ if there exists a highest weight vector $m_\Lambda \in M$ of weight Λ such that $M = \mathcal{U}_q(\widehat{\mathfrak{sl}_e}).m_\Lambda$. The same definition holds for $\mathcal{U}_q'(\widehat{\mathfrak{sl}_e})$-modules. From Theorem 3.1.4, one can deduce that every weight Λ' of a cyclic module M of highest weight Λ verifies $\Lambda - \Lambda' \in \sum_{i=0}^{e-1} \mathbb{Z}_{\geq 0}\alpha_i$, which justifies the terminology "highest weight". It also shows that the subspace M_Λ has dimension 1.

For $\Lambda \in P$, we can show that, up to isomorphism, there is a unique simple cyclic $\mathcal{U}_q(\widehat{\mathfrak{sl}_e})$-module of highest weight Λ, which we denote by $V(\Lambda)$. We further write $V'(\Lambda)$ for the restriction of $V(\Lambda)$ to $\mathcal{U}_q'(\widehat{\mathfrak{sl}_e})$. Then we have the following characterisation:

Proposition 3.1.6. *We have*

1. $V(\Lambda^{(1)}) \simeq V(\Lambda^{(2)})$ *if and only if* $\Lambda^{(1)} = \Lambda^{(2)}$.
2. $V'(\Lambda^{(1)}) \simeq V'(\Lambda^{(2)})$ *if and only if* $\Lambda^{(1)} - \Lambda^{(2)} \in \mathbb{Z}\delta$.

The category of integrable modules

Definition 3.1.7. A $\mathcal{U}_q(\widehat{\mathfrak{sl}_e})$-module M is called *integrable* if it verifies

1. $M = \bigoplus_{\Lambda \in P} M_\Lambda$,
2. For all $\Lambda \in P$, the vector space M_Λ is finite-dimensional,
3. For all $i \in [\![0, e-1]\!]$, for all $m \in M$, there exists $N \in \mathbb{Z}_{>0}$ such that $e_i^N.m = 0$ and $f_i^N.m = 0$.

The set of integrable $\mathcal{U}_q(\widehat{\mathfrak{sl}_e})$-modules is denoted $\mathcal{O}_{\text{int}}$.

The proposition below shows that it is sufficient to consider the case where Λ is a dominant weight, and that, in fact, the modules $V(\Lambda)$ with $\Lambda \in P^+$ give a complete set of non-isomorphic simple cyclic $\mathcal{U}_q(\widehat{\mathfrak{sl}_e})$-modules in $\mathcal{O}_{\text{int}}$. This is taken from [73, Proposition 3.2.8 and Theorem 3.5.4].

Proposition 3.1.8. *The category $\mathcal{O}_{\text{int}}$ has the following properties.*

- *For each $\Lambda \in P^+$, the module $V(\Lambda)$ is in $\mathcal{O}_{\text{int}}$,*
- *Each simple cyclic module in $\mathcal{O}_{\text{int}}$ is isomorphic to some $V(\Lambda)$ with $\Lambda \in P^+$.*
- *The category $\mathcal{O}_{\text{int}}$ is semisimple.*

Remark 3.1.9. This implies in particular that every integrable $\mathcal{U}_q(\widehat{\mathfrak{sl}_e})$-module is isomorphic to a direct sum of $V(\lambda)$'s.

3.1.4 Crystal basis, crystal graph, and canonical basis

The results in this section are mostly taken from the original paper [86], or from [131].

Kashiwara operators

Let M be an integrable $\mathcal{U}_q(\widehat{\mathfrak{sl}_e})$-module. Take $\Lambda \in P$, $i \in [\![0, e-1]\!]$ and $x \in M_\Lambda$. Set $N = \langle \Lambda, h_i \rangle$. First of all, one can prove that x writes uniquely as follows:

$$x = \sum_{k \geq \max(0,-N)} f_i^{(k)}.x_k,$$

with, for all k, $e_i.x_k = 0$, $t_i.x_k = q^{N+2k} x_k$, and only finitely many x_k are non-zero.

This gives rise to the definition:

Definition 3.1.10. The *Kashiwara operators* $\tilde{e}_i$ and $\tilde{f}_i$ are defined as follows:

$$\tilde{e}_i(x) = \sum_{k \geq \max(0,-N)} f_i^{(k-1)}.x_k \quad \text{and} \quad \tilde{f}_i(x) = \sum_{k \geq \max(0,-N)} f_i^{(k+1)}.x_k.$$

We can then extend the definition of the Kashiwara operators to the whole module M by linearity.

Remark 3.1.11. We will sometimes simply use the multiplicative notation for the action and the composition of the crystal operators, e.g. $\tilde{e}_i.x$, $\tilde{f}_i x$, $\tilde{e}_i \tilde{f}_j \tilde{f}_k x$.

Crystal bases

Now, let $\mathbb{A}$ be the ring of rational fractions which are regular at $q = 0$, that is

$$\mathbb{A} = \left\{ \frac{a(q)}{b(q)} \in \mathbb{Q}(q) \mid a(q), b(q) \in \mathbb{Q}[q] \quad \text{and} \quad b(0) \neq 0 \right\}.$$

Definition 3.1.12. Let M be an integrable $\mathcal{U}_q(\widehat{\mathfrak{sl}_e})$-module.

1. A *crystal lattice* for M is a free $\mathbb{A}$-module L such that:
   (a) $\mathbb{Q}(q) \otimes_{\mathbb{A}} L \simeq M$,
   (b) $L = \bigoplus_{\Lambda \in P} L_\Lambda$ where $L_\Lambda = L \cap M_\Lambda$,
   (c) $\tilde{e}_i.L \subset L$ and $\tilde{f}_i.L \subset L$.
2. A *crystal basis*[1] for M is a pair (L, C) such that:
   (a) L is a crystal lattice for M,
   (b) C is a basis of the $\mathbb{Q}(q)$-vector space L/qL,
   (c) $C = \bigsqcup_{\Lambda \in P} C_\Lambda$ where $C_\Lambda = C \cap (L_\Lambda / qL_\Lambda)$.
   (d) $\tilde{e}_i.C \subset C \sqcup \{0\}$ and $\tilde{f}_i.C \subset C \sqcup \{0\}$,
   (e) For all $u, v \in C$, we have $u = \tilde{e}_i(v)$ if and only if $v = \tilde{f}_i(u)$.

[1]In the original papers of Kashiwara [86], [87], the terminology "crystal base" is preferred.

Remark 3.1.13. Because of 1.(a), L is seen as an submodule of M, which ensures that 1.(b) makes sense. Similarly, we can see L_Λ/qL_Λ as a subspace of L/qL, which ensures that 2.(c) makes sense.

Since M is integrable, the elements e_i and f_i are locally nilpotent (Definition 3.1.7). This implies in particular that $\tilde{e}_i$ and $\tilde{f}_i$ are also locally nilpotent on L and therefore on $C \sqcup \{0\}$. For all $i \in [\![0, e-1]\!]$ and for all $u \in C$, we can then define

$$\varepsilon_i(u) = \max\{m \in \mathbb{Z}_{\geq 0} \mid \tilde{e}_i^m(u) \neq 0\}$$

and

$$\varphi_i(u) = \max\left\{m \in \mathbb{Z}_{\geq 0} \mid \tilde{f}_i^m(u) \neq 0\right\}.$$

The following result is taken from [86, Proposition 4.1, Theorems 4.2 and 4.3].

Theorem 3.1.14.

1. *Let $\Lambda \in P^+$. Consider the cyclic simple module $V(\Lambda)$, and denote by m_Λ the highest weight vector with weight Λ. Set*

$$L(\Lambda) = \sum \mathbb{A}\tilde{f}_{i_1} \ldots \tilde{f}_{i_k} m_\Lambda \quad \textit{and}$$

$$C(\Lambda) = \left\{\tilde{f}_{i_1} \ldots \tilde{f}_{i_k} m_\Lambda \mod qL(\Lambda)\right\} \backslash\{0\}.$$

   Then $(L(\Lambda), C(\Lambda))$ is a crystal basis for $V(\Lambda)$. Moreover, any crystal basis of $V(\Lambda)$ coincides with $(L(\Lambda), C(\Lambda))$ up to a scalar multiple.

2. *Let $\{M_i\}$ be a family of integrable $\mathcal{U}_q(\widehat{\mathfrak{sl}_e})$-modules, and let (L_i, C_i) be a crystal basis for M_i. Then $(\bigoplus_i L_i, \bigsqcup_i C_i)$ is a crystal basis for $\bigoplus_i M_i$.*

3. *Let $M \in \mathcal{O}_{\mathrm{int}}$, and let (L, C) be a crystal basis for M. Then there is an isomorphism $M \xrightarrow{\sim} \bigoplus_\Lambda V(\Lambda)$ which induces an isomorphism $(L, C) \xrightarrow{\sim} (\bigoplus_\Lambda L(\Lambda), \bigsqcup_\Lambda C(\Lambda))$.*

By Proposition 3.1.8, the first two points imply the existence (and construction) of a crystal basis for any integrable $\mathcal{U}_q(\widehat{\mathfrak{sl}_e})$-module, and the third point says it is unique up to isomorphism.

Tensor product rule

Because of the Hopf algebra structure of $\mathcal{U}_q(\widehat{\mathfrak{sl}_e})$, we have a simple combinatorial rule to determine the action of the Kashiwara operators on elementary tensors. The following result is taken from [73, Theorem 4.4.1].

Theorem 3.1.15. *Let M_1 and M_2 be two integrable $\mathcal{U}_q(\widehat{\mathfrak{sl}_e})$-modules, and for $i = 1, 2$, let (L_i, C_i) be a crystal basis of M_i. Set $L = L_1 \otimes_{\mathbb{A}} L_2$ and $C = C_1 \times C_2$. Then (L, C) is a crystal basis of $M_1 \otimes M_2$, and the action of the Kashiwara operators is given by*

- $\tilde{e}_i(u\otimes v) = \begin{cases} u\otimes \tilde{e}_i(v) & \textit{if } \varphi_i(v)\geq \varepsilon_i(u) \\ \tilde{e}_i(u)\otimes v & \textit{if } \varphi_i(v) < \varepsilon_i(u), \end{cases}$ *and*

- $\tilde{f}_i(u\otimes v) = \begin{cases} \tilde{f}_i(u)\otimes v & \textit{if } \varphi_i(v)\leq \varepsilon_i(u) \\ u\otimes \tilde{f}_i(v) & \textit{if } \varphi_i(v) > \varepsilon_i(u). \end{cases}$

This implies in particular that $\mathrm{wt}(u\otimes v) = \mathrm{wt}(u) + \mathrm{wt}(v)$.

Remark 3.1.16. Note that we have taken the reverse convention as in [73] for the coproduct on $\mathcal{U}_q(\widehat{\mathfrak{sl}_e})$, which gives a slightly different tensor product rule.

The crystal graph

Definition 3.1.17. Let M be an integrable $\mathcal{U}_q(\widehat{\mathfrak{sl}_e})$-module, and (L,C) be a crystal basis for M. The *crystal graph* B associated to (L,C) is the graph with

- vertices: the elements of C
- arrows: $u \xrightarrow{i} v$ for $u,v\in C$ and $i\in [\![0,e-1]\!]$ if and only if $v=\tilde{f}_i(u)$.

Remark 3.1.18. From now on, we will also sometimes simply call B the *crystal* associated to M.

Canonical bases for irreducible highest weight $\mathcal{U}_q(\widehat{\mathfrak{sl}_e})$-modules

We are ready to introduce the canonical (or global) basis of $V(\Lambda)$, which has been independently constructed by Kashiwara [86] and Lusztig [111].

Consider the $\mathbb{Q}[q,q^{-1}]$-algebra $\mathcal{U}_q(\widehat{\mathfrak{sl}_e})_{\mathbb{Q}} \subset \mathcal{U}_q(\widehat{\mathfrak{sl}_e})$ generated by the elements $e_i^{(k)}$, $f_i^{(k)}$, $\left\{ \begin{array}{c} t_i \\ k \end{array} \right\}$, t_i, where $k\in\mathbb{Z}_{\geq 0}$ and $i\in[\![0,e-1]\!]$. $\mathcal{U}_q(\widehat{\mathfrak{sl}_e})_{\mathbb{Q}}$ is called a $\mathbb{Q}$-*form*. For a $\mathcal{U}_q(\widehat{\mathfrak{sl}_e})$-module M, we denote by $M_{\mathbb{Q}}$ the corresponding $\mathcal{U}_q(\widehat{\mathfrak{sl}_e})_{\mathbb{Q}}$-module.

Define an involution on $\mathcal{U}_q(\widehat{\mathfrak{sl}_e})$ by

$$\overline{q}=q^{-1},\quad \overline{e_i}=e_i,\quad \overline{f_i}=f_i,\quad \overline{t_i}=t_i^{-1},\quad \overline{\mathfrak{d}}=\mathfrak{d}.$$

It induces an involution on $V(\Lambda)_{\mathbb{Q}}$, $\Lambda\in P^+$, by setting

$$\overline{u.m_\Lambda} = \overline{u}.m_\Lambda \qquad \text{for all } u\in\mathcal{U}_q(\widehat{\mathfrak{sl}_e})_{\mathbb{Q}}.$$

The following theorem claims that the basis $C(\Lambda)$ of $L(\Lambda)/qL(\Lambda)$ can be lifted to obtain a basis of $V(\Lambda)$.

Theorem 3.1.19. *Let $\Lambda\in P^+$, and $(L(\Lambda),C(\Lambda))$ be the crystal basis for $V(\Lambda)$. Then there exists a unique $\mathbb{Q}[q,q^{-1}]$-basis*

$$\mathcal{G}(\Lambda) = \{G(c) \mid c\in C(\Lambda)\}$$

of $V(\Lambda)_{\mathbb{Q}}$ such that, for all $c\in C(\Lambda)$,

1. *$G(c) = \overline{G(c)}$, and*

2. *$G(c) = c \mod qL(\Lambda)$.*

It is called the canonical basis (or the global crystal basis) of $V(\Lambda)$.

Remark 3.1.20. Actually, all the results presented in Sections 3.1.3 and 3.1.4 also make sense and hold for the algebra $\mathcal{U}'_q(\widehat{\mathfrak{sl}_e})$.

3.1.5 The limit case: the algebra $\mathcal{U}_q(\mathfrak{sl}_\infty)$

In this section, we focus on the algebra $\mathcal{U}_q(\mathfrak{sl}_\infty)$, which can be considered as the limit of the quantum algebras of type A_{e-1} as $e \longrightarrow \infty$. For more details about this algebra, the reader is invited to refer to [85]. For the purpose of this thesis, on can also take a look at [6] or [79].

The idea is to define the same objects as in the case of $\mathcal{U}_q(\widehat{\mathfrak{sl}_e})$, except that

1. their index set will be $\mathbb{Z}$ instead of $[\![0, e-1]\!]$ (and hence there is no notion of "modulo e" anymore),

2. we will forget about the element D, the weight δ and the generator $\mathfrak{d}$.

In this setting, we thus construct:

- The vector space $\mathfrak{h} = \bigoplus_{i\in\mathbb{Z}} \mathbb{Q}h_i$,

- The dual basis $\{\Lambda_i \; ; \; i \in \mathbb{Z}\}$ of the basis $\{h_i \; ; \; i \in \mathbb{Z}\}$, and the corresponding (dominant) weight lattice,

- The simple roots α_i and the root lattice,

- The $\mathbb{Q}(q)$-algebra $\mathcal{U}_q(\mathfrak{sl}_\infty)$ defined by its generators $e_i, f_i, t_i^{\pm 1}$ and the same relations as $\mathcal{U}_q(\widehat{\mathfrak{sl}_e})$.

We can then prove that Theorem 3.1.4 and Proposition 3.1.5 hold for $\mathcal{U}_q(\mathfrak{sl}_\infty)$. Moreover, the representation theory of $\mathcal{U}_q(\widehat{\mathfrak{sl}_e})$ exposed in Paragraph 3.1.3 is the same for $\mathcal{U}_q(\mathfrak{sl}_\infty)$, as well as the crystal theory exposed in Paragraph 3.1.4. More precisely, the construction of the crystal basis, crystal graph and canonical basis for irreducible highest weight $\mathcal{U}_q(\mathfrak{sl}_\infty)$-modules still make sense, and Theorems 3.1.14 and 3.1.19 still hold.

Notation 3.1.21. Since we use the same notations for the objects associated to $\mathcal{U}_q(\widehat{\mathfrak{sl}_e})$ and $\mathcal{U}_q(\mathfrak{sl}_\infty)$, we will use the subscript or superscript e or ∞ when the need to distinguish which quantum algebra is considered arises.

3.2 Fock spaces representations

This section is devoted to the study of the Fock space, a vector space which has both the structure of an integrable $\mathcal{U}_q(\widehat{\mathfrak{sl}_e})$ and $\mathcal{U}_q(\mathfrak{sl}_\infty)$-module. It gives a concrete realisation of the abstract modules $V(\Lambda)$ and the associated crystals.

From now on, we fix $e \in \mathbb{Z}_{>1} \sqcup \{\infty\}$, $l \in \mathbb{Z}_{>0}$ and $\mathbf{s} \in \mathbb{Z}^l$.

3.2.1 Definition and module structures

Definition 3.2.1. The Fock space $\mathcal{F}_{\mathbf{s}}$ is the $\mathbb{Q}(q)$-vector space with formal basis $|\boldsymbol{\lambda}, \mathbf{s}\rangle$ where $\boldsymbol{\lambda} \vdash_l n$ for all $n \in \mathbb{Z}_{>0}$, i.e.

$$\mathcal{F}_{\mathbf{s}} = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \bigoplus_{\boldsymbol{\lambda} \vdash_l n} \mathbb{Q}(q)|\boldsymbol{\lambda}, \mathbf{s}\rangle.$$

The integer l is called the *level* of $\mathcal{F}_{\mathbf{s}}$.

We define an order on the set of addable or removable i-node of a l-partition $\boldsymbol{\lambda}$. Let $\gamma = (a, b, c)$ and $\gamma' = (a', b', c')$ be two removable or addable i-nodes of $\boldsymbol{\lambda} \vdash_l n$. We write

$$\gamma \prec_{\mathbf{s}} \gamma' \text{ if } \begin{cases} b - a + s_c < b' - a' + s_{c'} \text{ or} \\ b - a + s_c = b' - a' + s_{c'} \quad \text{and} \quad c > c'. \end{cases}$$

Let $\boldsymbol{\lambda} \vdash_l n$ and $\boldsymbol{\mu} \vdash_l n+1$ such that $[\boldsymbol{\mu}] = [\boldsymbol{\lambda}] \cup \{\gamma\}$ where γ is an i-node. We set

$$\begin{aligned} N_i^{\prec}(\boldsymbol{\lambda}, \boldsymbol{\mu}) =& \sharp \left\{ \text{addable } i\text{-nodes } \gamma' \text{ of } \boldsymbol{\lambda} \text{ such that } \gamma' \prec_{\mathbf{s}} \gamma \right\} - \\ & \sharp \left\{ \text{removable } i\text{-nodes } \gamma' \text{ of } \boldsymbol{\mu} \text{ such that } \gamma' \prec_{\mathbf{s}} \gamma \right\}, \end{aligned}$$

$$\begin{aligned} N_i^{\succ}(\boldsymbol{\lambda}, \boldsymbol{\mu}) =& \sharp \left\{ \text{addable } i\text{-nodes } \gamma' \text{ of } \boldsymbol{\lambda} \text{ such that } \gamma' \succ_{\mathbf{s}} \gamma \right\} - \\ & \sharp \left\{ \text{removable } i\text{-nodes } \gamma' \text{ of } \boldsymbol{\mu} \text{ such that } \gamma' \succ_{\mathbf{s}} \gamma \right\}, \end{aligned}$$

$$N_i(\boldsymbol{\lambda}) = \sharp \left\{ \text{addable } i\text{-nodes of } \boldsymbol{\lambda} \right\} - \sharp \left\{ \text{removable } i\text{-nodes of } \boldsymbol{\lambda} \right\},$$

$$\text{and} \qquad N_{\mathfrak{d}}(\boldsymbol{\lambda}) = \sharp \left\{ 0\text{-nodes of } \boldsymbol{\lambda} \right\}.$$

The following result is due to Jimbo, Misra, Miwa and Okado [82].

Theorem 3.2.2. *Let $\boldsymbol{\lambda} \vdash_l n$. The formulas*

$$\begin{aligned} e_i.\, |\boldsymbol{\lambda}, \mathbf{s}\rangle &= \sum_{\substack{\boldsymbol{\mu} \vdash_l n-1 \\ \mathrm{res}([\boldsymbol{\lambda}] \backslash [\boldsymbol{\mu}]) = i}} q^{-N_i^{\prec}(\boldsymbol{\mu}, \boldsymbol{\lambda})}\, |\boldsymbol{\mu}, \mathbf{s}\rangle, \\ f_i.\, |\boldsymbol{\lambda}, \mathbf{s}\rangle &= \sum_{\substack{\boldsymbol{\mu} \vdash_l n-1 \\ \mathrm{res}([\boldsymbol{\mu}] \backslash [\boldsymbol{\lambda}]) = i}} q^{-N_i^{\succ}(\boldsymbol{\lambda}, \boldsymbol{\mu})}\, |\boldsymbol{\mu}, \mathbf{s}\rangle, \\ t_i.|\boldsymbol{\lambda}, \mathbf{s}\rangle &= q^{N_i(\boldsymbol{\lambda})}|\boldsymbol{\lambda}, \mathbf{s}\rangle \quad \textit{and} \\ \mathfrak{d}.|\boldsymbol{\lambda}, \mathbf{s}\rangle &= -(\Delta(\mathbf{s}, n) + N_{\mathfrak{d}}(\boldsymbol{\lambda}))|\boldsymbol{\lambda}, \mathbf{s}\rangle, \ \textit{for all } i \in [\![0, e-1]\!], \end{aligned}$$

where

$$\Delta(\mathbf{s},n)=\frac{1}{2}\sum_{c=1}^{l}\left(\frac{s_c^2}{n}-s_c\right)-\left(\frac{(s_c \mod n)^2}{n}-(s_c \mod n)\right)$$

endow $\mathcal{F}_{\mathbf{s}}$ with the structure of an integrable $\mathcal{U}_q(\widehat{\mathfrak{sl}_e})$-module.

Remark 3.2.3. Accordingly, the first three formulas endow $\mathcal{F}_{\mathbf{s}}$ with the structure of an integrable $\mathcal{U}'_q(\widehat{\mathfrak{sl}_e})$-module (see Remark 3.1.20). Besides, because of the definition of $\mathcal{U}_q(\mathfrak{sl}_\infty)$, see Section 3.1.5, these three formulas also endow $\mathcal{F}_{\mathbf{s}}$ with the structure of an integrable $\mathcal{U}_q(\mathfrak{sl}_\infty)$-module, provided we replace $\mathrm{res}([\boldsymbol{\lambda}]\backslash[\boldsymbol{\mu}])$ by $\mathrm{cont}([\boldsymbol{\lambda}]\backslash[\boldsymbol{\mu}])$ (since in the $\mathcal{U}_q(\mathfrak{sl}_\infty)$, the notion of residue does not make sense and is replaced by the notion of content).

These are actually the two module structures we will be interested in, the action of $\mathfrak{d}$ being not really important for our purpose.

Remark 3.2.4. According to Remark 3.1.3, taking $q=1$, the Fock space is endowed with the structure of an integrable $\mathcal{U}(\widehat{\mathfrak{sl}_e})$-module (resp. $\mathcal{U}(\mathfrak{sl}_\infty)$-module).

Proposition 3.2.5. *We have*

$$\mathrm{wt}(|\boldsymbol{\lambda},\mathbf{s}\rangle)=-\Delta(\mathbf{s},n)\delta+\Lambda_{s_1}+\cdots+\Lambda_{s_l}-\sum_{i=0}^{e-1}N_i(\boldsymbol{\lambda})\alpha_i.$$

Therefore, as an element of the $\mathcal{U}'_q(\widehat{\mathfrak{sl}_e})$-module (resp. $\mathcal{U}_q(\mathfrak{sl}_\infty)$-module) $\mathcal{F}_{\mathbf{s}}$, we have

$$\mathrm{wt}(|\boldsymbol{\lambda},\mathbf{s}\rangle)=\Lambda_{s_1}+\cdots+\Lambda_{s_l}-\sum_{i=0}^{e-1}N_i(\boldsymbol{\lambda})\alpha_i.$$

Recall that in the case of $\mathcal{U}_q(\widehat{\mathfrak{sl}_e})$ and $\mathcal{U}'_q(\widehat{\mathfrak{sl}_e})$, one must read the indices modulo e.

3.2.2 Crystal structure on the Fock space

From now on, when $e<\infty$, we will only consider the $\mathcal{U}'_q(\widehat{\mathfrak{sl}_e})$-module structure on $\mathcal{F}_{\mathbf{s}}$, following Remark 3.2.3. This means that we will no longer take into account the weight δ nor the generator $\mathfrak{d}$. By Remark 3.1.20, it makes sense to study the crystal structure on $\mathcal{F}_{\mathbf{s}}$ as a $\mathcal{U}'_q(\widehat{\mathfrak{sl}_e})$-module, which is the aim of this section.

A realisation of the irreducible highest weight modules

One checks that the element $|\boldsymbol{\emptyset},\mathbf{s}\rangle\in\mathcal{F}_{\mathbf{s}}$ is a highest weight vector, of weight $\Lambda_{\mathbf{s}}=\Lambda_{s_1}+\cdots+\Lambda_{s_l}$. We denote

$$V(\mathbf{s})=\mathcal{U}'_q(\widehat{\mathfrak{sl}_e}).|\boldsymbol{\emptyset},\mathbf{s}\rangle, \tag{3.4}$$

that is, the submodule of $\mathcal{F}_{\mathbf{s}}$ spanned by $|\boldsymbol{\emptyset},\mathbf{s}\rangle$. Because $|\boldsymbol{\emptyset},\mathbf{s}\rangle$ is a highest weight vector, $V(\mathbf{s})$ is an irreducible highest weight $\mathcal{U}'_q(\widehat{\mathfrak{sl}_e})$-module. By Proposition 3.1.6, $V(\mathbf{s})$ is isomorphic to $V'(\mathrm{wt}(|\boldsymbol{\emptyset},\mathbf{s}\rangle))=V'(\Lambda_{\mathbf{s}})$. We deduce the following important property.

Proposition 3.2.6. *Let* $\mathbf{s} = (s_1, \dots, s_l)$ *and* $\mathbf{r} = (r_1, \dots, r_l)$ *be two* l*-charges. Then* $V(\mathbf{s}) \simeq V(\mathbf{r})$ *if and only if* $\mathbf{s}$ *and* $\mathbf{r}$ *are in the same orbit under the action of* $\widehat{\mathfrak{S}_l}$*.*

Proof. Denote $\Lambda_{\mathbf{s}} = \Lambda_{s_1} + \cdots + \Lambda_{s_l}$ and $\Lambda_{\mathbf{r}} = \Lambda_{r_1} + \cdots + \Lambda_{r_l}$. Then $V(\mathbf{s}) \simeq V'(\Lambda_{\mathbf{s}})$ and $V(\mathbf{r}) \simeq V'(\Lambda_{\mathbf{r}})$. By Proposition 3.1.6, $V'(\Lambda_{\mathbf{s}}) \simeq V'(\Lambda_{\mathbf{r}})$ if and only if $\Lambda_{\mathbf{s}} = \Lambda_{\mathbf{r}}$. Recall that $\widehat{\mathfrak{S}_l}$ acts on $\mathbb{Z}^l$ by permuting and translating by a multiple of e the components of an l-charge. The indices in $\Lambda_{\mathbf{s}}$ and $\Lambda_{\mathbf{r}}$ being read modulo e, the result follows. □

By Theorem 3.1.14, one can consider the crystal basis for $V(\mathbf{s})$. The construction of the crystal graph, which we denote by $B(\mathbf{s})$, and the canonical basis $\mathcal{G}(\mathbf{s})$, then follows. However, there is another approach to construct the crystal and canonical basis for $V(\mathbf{s})$. In fact, the whole Fock space has a crystal basis since it is an integrable $\mathcal{U}_q'(\widehat{\mathfrak{sl}_e})$-module. It turns out that we can directly construct this crystal basis, and that it induces the construction of the crystal basis for the irreducible highest weight submodule $V(\mathbf{s})$. This is what we detail below, and is inspired from the work of Jimbo, Misra, Miwa and Okado [82].

Crystal of the Fock space

As in Paragraph 3.1.4, let

$$\mathbb{A} = \left\{ \frac{a(q)}{b(q)} \in \mathbb{Q}(q) \mid a(q), b(q) \in \mathbb{Q}[q] \quad \text{and} \quad b(0) \neq 0 \right\},$$

the ring of rational fractions which are regular at $q = 0$. Define

$$L(\mathcal{F}_{\mathbf{s}}) = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \bigoplus_{\boldsymbol{\lambda} \vdash_l n} \mathbb{A} |\boldsymbol{\lambda}, \mathbf{s}\rangle \quad \text{and}$$

$$C(\mathcal{F}_{\mathbf{s}}) = \{ |\boldsymbol{\lambda}, \mathbf{s}\rangle \mod qL \ ; \ \boldsymbol{\lambda} \vdash_l n, \, n \in \mathbb{Z}_{\geq 0} \}.$$

Theorem 3.2.7. *The pair* $(L(\mathcal{F}_{\mathbf{s}}), C(\mathcal{F}_{\mathbf{s}}))$ *is a crystal basis for* $\mathcal{F}_{\mathbf{s}}$ *considered as a* $\mathcal{U}_q'(\widehat{\mathfrak{sl}_e})$*-module (resp.* $\mathcal{U}_q(\mathfrak{sl}_\infty)$*-module).*

According to Definition 3.1.17, the $\mathcal{U}_q'(\widehat{\mathfrak{sl}_e})$-module (resp. $\mathcal{U}_q(\mathfrak{sl}_\infty)$-module) $\mathcal{F}_{\mathbf{s}}$ is then equipped with a crystal graph, which we denote by $B_e(\mathcal{F}_{\mathbf{s}})$ (resp. $B_\infty(\mathcal{F}_{\mathbf{s}})$), in accordance with Notation 3.1.21. Note that even if the crystal basis is the same for both module structures, this does not mean that the crystal structure (i.e. the crystal graphs $B_e(\mathcal{F}_{\mathbf{s}})$ and $B_\infty(\mathcal{F}_{\mathbf{s}})$) coincide. In fact, they are different, as claimed in Theorem 3.2.10 below.

In order to determine the crystal graph $B(\mathcal{F}_{\mathbf{s}})$, we first need to introduce the notion of *good* nodes. Let $\boldsymbol{\lambda} \vdash_l n$. Consider the set of its addable and removable i-nodes, ordered with respect to $\prec_{\mathbf{s}}$. Encode each addable (resp. removable) i-node with the letter A (resp. R). This yields a word of the form $A^{\alpha_1} R^{\beta_1} \dots A^{\alpha_p} R^{\beta_p}$. Delete recursively all the occurences of type RA in this word. We get a word of the form $A^\alpha R^\beta$. Denote it by $w_i(\boldsymbol{\lambda})$. Let γ be the rightmost addable (resp. leftmost removable) i-node in $w_i(\boldsymbol{\lambda})$.

Definition 3.2.8. The node γ is called the *good addable* (resp. *good removable*) i-node of $\boldsymbol{\lambda}$.

Remark 3.2.9. Clearly, the good addable (resp. removable) i-nodes are different in general when e varies, because the words $w_i(\boldsymbol{\lambda})$ are different. In particular, for $e < \infty$ fixed, $w_i^e(\boldsymbol{\lambda}) \neq w_i^\infty(\boldsymbol{\lambda})$, and the good addable (resp. removable) i-node of $\boldsymbol{\lambda}$ for $e < \infty$ is different from the good addable (resp. removable) i-node of $\boldsymbol{\lambda}$ for $e = \infty$.

Theorem 3.2.10 ([82],[126])**.** *the crystal graph $B(\mathcal{F}_\mathbf{s})$ consists of:*

- *vertices: all the multipartitions $\boldsymbol{\lambda} \vdash_l n$ for $n \in \mathbb{Z}_{\geq 0}$,*
- *arrows: $\boldsymbol{\lambda} \overset{i}{\longrightarrow} \boldsymbol{\mu}$ if and only if $[\boldsymbol{\mu}] = [\boldsymbol{\lambda}] \cup \{\gamma\}$ where γ is the good addable i-node of $\boldsymbol{\lambda}$.*

Remark 3.2.11. Because of Remark 3.2.9, we see that the $\mathcal{U}_q'(\widehat{\mathfrak{sl}_e})$-crystal structure and the $\mathcal{U}_q(\mathfrak{sl}_\infty)$-crystal structure do not coincide.

Crystal of the irreducible highest weight modules

The crystal $B(\mathcal{F}_\mathbf{s})$ has several connected components. One can look at the connected component containing the empty l-partition $\boldsymbol{\emptyset}$. By Theorem 3.1.14, this connected component can be identified with $B(\mathbf{s})$, the crystal graph of $V(\mathbf{s}) = \mathcal{U}_q'(\widehat{\mathfrak{sl}_e}).|\boldsymbol{\emptyset}, \mathbf{s}\rangle$. The vertices of $B(\mathbf{s})$ are called the *Uglov l-partitions*, and denoted by $\Phi_\mathbf{s}$. We further denote $\Phi_\mathbf{s}(n)$ the Uglov l-partitions of rank n. Because $|\boldsymbol{\emptyset}, \mathbf{s}\rangle$ is a highest weight vector in $\mathcal{F}_\mathbf{s}$, $\boldsymbol{\emptyset}$ is a *highest weight vertex* in $B(\mathbf{s})$, that is, it has zero-indegree. Besides, because of Theorem 3.2.10, it is straightforward that the set of Uglov l-partitions has the following recursive characterisation.

Corollary 3.2.12. *The set $\Phi_\mathbf{s}$ is characterised by:*

- *$\boldsymbol{\emptyset} \in \Phi_\mathbf{s}$,*
- *If $\boldsymbol{\mu} \in \Phi_\mathbf{s}$, then any $\boldsymbol{\lambda}$ obtained from $\boldsymbol{\mu}$ by adding a good addable node is also in $\Phi_\mathbf{s}$.*

Remark 3.2.13. In particular, the canonical basis $\mathcal{G}(\mathbf{s})$ of $V(\mathbf{s})$ is labelled by the Uglov l-partitions. We write $\mathcal{G}(\mathbf{s}) = \{G(\boldsymbol{\mu}, \mathbf{s}) \; ; \; \boldsymbol{\mu} \in \Phi_\mathbf{s}\}$.

3.2.3 Uglov's canonical basis of the Fock space

In this section, we roughly explain how Uglov constructed in [126] a basis for the whole Fock space, also called the "canonical basis". From this basis, we will recover the canonical basis $\mathcal{G}(\mathbf{s})$ of the irreducible highest weight module $V(\mathbf{s})$.

Canonical basis in the sense of Uglov

In order to do that, Uglov starts by defining an involution $\overline{}$ on the q-wedge product Λ^r, for $r \in \mathbb{Z}_{>0}$. He shows ([126, Theorem 3.25]) that this yields a unique "canonical basis" of Λ^r, in the sense that it verifies the relations of Theorem 3.1.19. Then, for each s, he constructs an inductive limit $\Lambda^{s+\frac{\infty}{2}} = \varinjlim \Lambda^r$, called the semi-infinite q-wedge product. This space is therefore endowed with the involution induced by that of Λ^s. It turns out that $\Lambda^{s+\frac{\infty}{2}}$ has the structure of a $\mathcal{U}'_q(\widehat{\mathfrak{sl}_e})$-module (resp. $\mathcal{U}_q(\mathfrak{sl}_\infty)$-module), and that the module $\mathcal{F}_\mathbf{s}$ can be embedded in $\Lambda^{s+\frac{\infty}{2}}$. This yields an involution on $\mathcal{F}_\mathbf{s}$, compatible with the action of the quantum algebra, and the construction of a basis $\mathcal{G}(\mathcal{F}_\mathbf{s})$ which verifies the same relations, as stated in the theorem below. Set

$$\mathcal{L}(\mathcal{F}_\mathbf{s}) = \bigoplus_{n\in\mathbb{Z}_{\geq 0}} \bigoplus_{\boldsymbol{\lambda}\vdash_l n} \mathbb{Z}[q]|\boldsymbol{\lambda},\mathbf{s}\rangle.$$

Theorem 3.2.14. *There exists a unique basis*

$$\mathcal{G}(\mathcal{F}_\mathbf{s}) = \{\mathbf{G}(\boldsymbol{\lambda},\mathbf{s})\ ;\ \boldsymbol{\lambda}\vdash_l n,\, n\in\mathbb{Z}_{\geq 0}\}$$

of $\mathcal{F}_\mathbf{s}$ such that, for all $n \in \mathbb{Z}_{\geq 0}$ and $\boldsymbol{\lambda}\vdash_l n$,

1. $\overline{\mathbf{G}(\boldsymbol{\lambda},\mathbf{s})} = \mathbf{G}(\boldsymbol{\lambda},\mathbf{s})$, *and*
2. $\mathbf{G}(\boldsymbol{\lambda},\mathbf{s}) = \boldsymbol{\lambda} \mod q\mathcal{L}(\mathbf{F_s})$.

It is called the canonical basis of $\mathcal{F}_\mathbf{s}$.

Remark 3.2.15. Note that, again, this holds for both the $\mathcal{U}'_q(\widehat{\mathfrak{sl}_e})$-module and the $\mathcal{U}_q(\mathfrak{sl}_\infty)$-module structure on $\mathcal{F}_\mathbf{s}$.

Compatibility with Kashiwara's canonical basis

The terminology "canonical basis" for $\mathcal{F}_\mathbf{s}$ is justified by the fact that, on the one hand, the elements $\mathbf{G}(\boldsymbol{\lambda},\mathbf{s})$ and $G(\boldsymbol{\mu},\mathbf{s})$ (cf. Theorem 3.1.19 together with Remark 3.2.13) verify the same relations; and, on the other hand, both notions are compatible in the following sense. Define

$$\mathcal{G}'(\mathbf{s}) = \mathcal{G}(\mathcal{F}_\mathbf{s}) \cap V(\mathbf{s}).$$

In other words, by definition of $\Phi_\mathbf{s}$,

$$\mathcal{G}'(\mathbf{s}) = \{\mathbf{G}(\boldsymbol{\mu},\mathbf{s})\ ;\ \boldsymbol{\mu}\in\Phi_\mathbf{s}\}.$$

Recall that we have denoted $\mathcal{G}(\mathbf{s})$ the canonical basis of $V(\mathbf{s})$.

Theorem 3.2.16.

$$\mathcal{G}'(\mathbf{s}) = \mathcal{G}(\mathbf{s}).$$

This means that one recovers the canonical basis of $V(\mathbf{s})$ by keeping the elements of $\mathcal{G}(\mathcal{F}_\mathbf{s})$ which are labelled by the Uglov l-partitions.

3.2.4 Dual canonical basis as perfect basis

The dual canonical basis

Recall that we have defined a scalar product $(.|.)$ in Section 3.1.1 via the formulas (3.3). For $\boldsymbol{\lambda} \vdash_l n$, we set

$$||\boldsymbol{\lambda}|| = \frac{1}{2}\left(\mathrm{wt}(\boldsymbol{\lambda})|\mathrm{wt}(\boldsymbol{\lambda})\right).$$

One can then define a scalar product $(.,.)$ on $\mathcal{F}_{\mathbf{s}}$ by setting

$$(\boldsymbol{\lambda},\boldsymbol{\mu}) = q^{||\boldsymbol{\lambda}||}\delta_{\boldsymbol{\lambda},\boldsymbol{\mu}},$$

for all $\boldsymbol{\lambda},\boldsymbol{\mu} \vdash_l n$, for all $n \in \mathbb{Z}_{\geq 0}$.

Proposition 3.2.17. *let $u, v \in \mathcal{F}_{\mathbf{s}}$ and $i \in [\![0, e-1]\!]$. Then*

$$(t_i.u, v) = (u, t_i.v) \qquad \text{and} \qquad (e_i.u, v) = (u, f_i.v).$$

Now, let

$$\mathcal{G}^*(\mathcal{F}_{\mathbf{s}}) = \{\mathbf{G}^*(\boldsymbol{\lambda}, \mathbf{s})\ ;\ \boldsymbol{\lambda} \vdash_l n,\ n \in \mathbb{Z}_{\geq 0}\}$$

be the basis of $\mathcal{F}_{\mathbf{s}}$ which is adjoint to the canonical basis $\mathcal{G}(\mathcal{F}_{\mathbf{s}})$ with respect to $(.,.)$, i.e.

$$(\mathbf{G}(\boldsymbol{\lambda},\mathbf{s}), \mathbf{G}^*(\boldsymbol{\mu},\mathbf{s})) = \delta_{\boldsymbol{\lambda},\boldsymbol{\mu}}, \tag{3.5}$$

for all $n \in \mathbb{Z}_{\geq 0}$ and $\boldsymbol{\lambda},\boldsymbol{\mu} \vdash_l n$. We call $\mathcal{G}^*(\mathcal{F}_{\mathbf{s}})$ the *dual canonical basis* of $\mathcal{F}_{\mathbf{s}}$.

Perfect bases

Berenstein and Kazhdan introduced in [9] the notion of perfect bases as an "unquantized" version of Kashiwara's crystal bases. In fact, such a basis defines a "crystal" for the $\mathcal{U}(\widehat{\mathfrak{sl}_e})$-module $\mathcal{F}_{\mathbf{s}}$ (cf. Remark 3.2.4), in the sense that it induces similar properties as that of Definition 3.1.12. In this pararagraph, we will work at $q = 1$.

Let V be an integrable $\mathcal{U}(\widehat{\mathfrak{sl}_e})$-module. For $i \in [\![0, e-1]\!]$ and $v \in V$, set

$$l_i(v) = \max\{m \in \mathbb{Z}_{\geq 0} \mid e_i^m.v \neq 0\}.$$

For $m \in \mathbb{Z}_{\geq 0}$, denote

$$V_i^{<m} = \{v \in V \mid l_i(v) < m\}.$$

Let B be a basis of V consisting of weight vectors.

Definition 3.2.18. The basis B is called a *perfect* if it is equipped with maps $\hat{e}_i, \hat{f}_i : B \longrightarrow B \sqcup \{0\}$, for $i \in [\![0, e-1]\!]$, verifying

1. for $b, b' \in B$, we have $\hat{f}_i(b) = b'$ if and only if $\hat{e}_i(b') = b$,
2. $\hat{e}_i(b) \neq 0$ if and only if $e_i.b \neq 0$, and

3. if $e_i(b) \neq 0$, then
$$e_i(b) \in \mathbb{C}^\times \hat{e}_i(b) + V_i^{<l_i(b)-1}. \tag{3.6}$$

Consider the dual canonical basis of $\mathcal{F}_{\mathbf{s}}$ defined in (3.5). Equip it with $\hat{e}_i$ and $\hat{f}_i$ defined by
$$\hat{e}_i(\mathbf{G}^*(\boldsymbol{\lambda},\mathbf{s})) = \mathbf{G}^*(\tilde{e}_i(\boldsymbol{\lambda}),\mathbf{s}) \qquad \text{and} \qquad \hat{f}_i(\mathbf{G}^*(\boldsymbol{\lambda},\mathbf{s})) = \mathbf{G}^*(\tilde{f}_i(\boldsymbol{\lambda}),\mathbf{s}),$$
where $\tilde{e}_i$ and $\tilde{f}_i$ are the Kashiwara operators defined in Section 3.2.2. Set also $\hat{e}_i(\mathcal{G}^*(\boldsymbol{\lambda},\mathbf{s})) = 0$ (resp. $\hat{f}_i(\mathcal{G}^*(\boldsymbol{\lambda},\mathbf{s})) = 0$) whenever $\tilde{e}_i(\boldsymbol{\lambda}) = 0$ (resp. $\tilde{f}_i(\boldsymbol{\lambda}) = 0$).

Denote $\mathcal{G}_1^*(\mathcal{F}_{\mathbf{s}}) = \{\mathbf{G}_1^*(\boldsymbol{\lambda},\mathbf{s})\ ;\ \boldsymbol{\lambda} \vdash_l n,\, n \in \mathbb{Z}_{\geq 0}\}$ the dual canonical basis of $\mathcal{F}_{\mathbf{s}}$ specialised at $q = 1$. The following result is needed in the proof of Theorem 3.3.4 in the upcoming section.

Proposition 3.2.19. *The basis $\mathcal{G}_1^*(\mathcal{F}_{\mathbf{s}})$ is perfect.*

Proof. Let us prove that the three points of Definition 3.2.18 hold.

1. This directly comes from Point 2.(e) of Definition 3.1.12.
$$\begin{aligned} \hat{f}_i(\mathbf{G}^*(\boldsymbol{\lambda},\mathbf{s})) = \mathbf{G}^*(\boldsymbol{\mu},\mathbf{s}) \quad &\Leftrightarrow \quad \mathbf{G}^*(\tilde{f}_i(\boldsymbol{\lambda}),\mathbf{s}) = \mathbf{G}^*(\boldsymbol{\mu},\mathbf{s}) \\ &\Leftrightarrow \quad \mathbf{G}^*(\boldsymbol{\lambda},\mathbf{s}) = \mathbf{G}^*(\tilde{e}_i(\boldsymbol{\mu}),\mathbf{s}) \\ &\Leftrightarrow \quad \mathbf{G}^*(\boldsymbol{\lambda},\mathbf{s}) = \hat{e}_i(\mathbf{G}^*(\boldsymbol{\mu},\mathbf{s})). \end{aligned}$$

   By specialising q at 1, we get the expected relation.

Now, by [88, Lemma 12.1], we know that the canonical basis $\mathcal{G}(\mathcal{F}_{\mathbf{s}}) = \{\mathbf{G}(\boldsymbol{\lambda},\mathbf{s})\}$ verifies
$$f_i.\mathbf{G}(\boldsymbol{\lambda},\mathbf{s}) = [\varepsilon_i(\boldsymbol{\lambda})+1]\mathbf{G}(\tilde{f}_i(\boldsymbol{\lambda})) + \sum_{\varepsilon_j(\boldsymbol{\mu}) \geq \varepsilon_j(\boldsymbol{\lambda}) + \langle \alpha_i, h_j\rangle} c_{\boldsymbol{\lambda},\boldsymbol{\mu}} \mathbf{G}(\boldsymbol{\mu},\mathbf{s}), \tag{3.7}$$
for all $i, j \in [\![0, e-1]\!]$ and for all l-partitions $\boldsymbol{\lambda}$, where $c_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ are some coefficients in $\mathbb{Q}(q)$. For a proof of this statement, we refer to either [87, Proposition 5.3.1] or [89, Proposition 6.2.3]. In particular, take $i = j$. We have $\langle \alpha_i, h_i\rangle = 2$ by definition of α_i (3.1).

Write
$$e_i.\mathbf{G}^*(\boldsymbol{\lambda},\mathbf{s}) = \sum_{\boldsymbol{\mu} \vdash_l |\boldsymbol{\lambda}|-1} b_{\boldsymbol{\lambda},\boldsymbol{\mu}} \mathbf{G}^*(\boldsymbol{\mu},\mathbf{s}). \tag{3.8}$$
We compute the coefficients $b_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ as follows:
$$\begin{aligned} b_{\boldsymbol{\lambda},\boldsymbol{\mu}} &= (e_i.\mathbf{G}^*(\boldsymbol{\lambda},\mathbf{s}), \mathbf{G}(\boldsymbol{\mu},\mathbf{s})) && \text{by 3.5} \\ &= (\mathbf{G}^*(\boldsymbol{\lambda},\mathbf{s}), f_i.\mathbf{G}(\boldsymbol{\mu},\mathbf{s})) && \text{by Proposition 3.2.17} \\ &= (\mathbf{G}^*(\boldsymbol{\lambda},\mathbf{s}), [\varepsilon_i(\boldsymbol{\mu})+1]\mathbf{G}(\tilde{f}_i(\boldsymbol{\mu}),\mathbf{s}) + \\ &\qquad \sum_{\varepsilon_i(\boldsymbol{\nu}) \geq \varepsilon_i(\boldsymbol{\mu})+2} c_{\boldsymbol{\mu},\boldsymbol{\nu}} \mathbf{G}(\boldsymbol{\nu},\mathbf{s})) && \text{by (3.7)} \\ &= \begin{cases} [\varepsilon_i(\boldsymbol{\mu})+1] = [\varepsilon(\boldsymbol{\lambda})] & \text{if } \boldsymbol{\mu} = \tilde{e}_i(\boldsymbol{\lambda}) \\ c_{\boldsymbol{\mu},\boldsymbol{\lambda}} \text{ s.t. } \varepsilon_i(\boldsymbol{\mu}) \leq \varepsilon_i(\boldsymbol{\lambda}) - 2 & \text{if } \boldsymbol{\mu} \neq \tilde{e}_i(\boldsymbol{\lambda}) \end{cases} && \text{by (3.5)} \end{aligned}$$

Therefore, we can rewrite the expansion (3.8) as

$$e_i.\mathbf{G}^*(\boldsymbol{\lambda}, \mathbf{s}) = [\varepsilon_i(\boldsymbol{\lambda})]\hat{e}_i(\mathbf{G}^*(\boldsymbol{\lambda}, \mathbf{s})) + \sum_{\varepsilon_i(\boldsymbol{\mu})<\varepsilon_i(\boldsymbol{\lambda})-1} c_{\boldsymbol{\mu},\boldsymbol{\lambda}}\mathbf{G}^*(\boldsymbol{\mu}, \mathbf{s}). \tag{3.9}$$

Using this formula, we can now prove the two remaining points.

2. Suppose $\hat{e}_i(\mathbf{G}^*(\boldsymbol{\lambda}, \mathbf{s})) = 0$, i.e. $\mathbf{G}^*(\tilde{e}_i(\boldsymbol{\lambda}), \mathbf{s}) = 0$. Then $\tilde{e}_i(\boldsymbol{\lambda}) = 0$, and $\varepsilon_i(\boldsymbol{\lambda}) = 0$. Since all $\boldsymbol{\mu}$ appearing in $e_i.\mathbf{G}^*(\boldsymbol{\lambda}, \mathbf{s})$ verify $\varepsilon_i(\boldsymbol{\mu}) < \varepsilon_i(\boldsymbol{\lambda}) - 1$, the sum is empty. Therefore $e_i.\mathbf{G}^*(\boldsymbol{\lambda}, \mathbf{s}) = 0$. In particular, this is true at $q = 1$.
3. If we take $q = 1$ in (3.9), we first see that the coefficient of $\hat{e}_i(\mathbf{G}_1^*(\boldsymbol{\lambda}, \mathbf{s}))$ is $\varepsilon_i(\boldsymbol{\lambda})$. If $\hat{e}_i(\mathbf{G}_1^*(\boldsymbol{\lambda}, \mathbf{s})) \neq 0$, then $\tilde{e}_i(\boldsymbol{\lambda}) \neq 0$, and $\varepsilon_i(\boldsymbol{\lambda}) \neq 0$. We also have that $l_i(\mathbf{G}_1^*(\boldsymbol{\mu}, \mathbf{s})) = \varepsilon_i(\boldsymbol{\mu})$ for all $\boldsymbol{\mu}$. Therefore, $\varepsilon_i(\boldsymbol{\mu}) < \varepsilon_i(\boldsymbol{\lambda}) - 1 \Leftrightarrow l_i(\mathbf{G}_1^*(\boldsymbol{\mu}, \mathbf{s})) < l_i(\mathbf{G}_1^*(\boldsymbol{\lambda}, \mathbf{s})) - 1$, which shows that the remaining terms in the sum are in $(\mathcal{F}_\mathbf{s})_i^{<l_i(\mathbf{G}_1^*(\boldsymbol{\lambda},\mathbf{s}))-1}$.

□

Remark 3.2.20. In [79], a characterisation of the highest weight vertices in $B(\mathcal{F}_\mathbf{s})$ is given. This gives a way to determine highest weight vectors, provided we know the dual canonical basis. For examples of computations of the dual canonical basis of $\mathcal{F}_\mathbf{s}$, see [131].

3.3 Links with the representation theory of $G(l, 1, n)$

The representation theory of the affine quantum algebra of type A turns out to be intimately related to the modular representation theory of the complex reflection group $W_n = G(l, 1, n)$, as developped in the works of authors such as Lascoux-Leclerc-Thibon [95], Kleshchev [92], [93], [94], Ariki [2], Grojnowski [64] Brundan-Kleshchev [20] and Chuang-Rouquier [30]. One of the most significant results is Ariki's proof of the LLT conjecture, which is exposed in the upcoming paragraph. It enables the computation of decomposition matrices for Hecke algebras of W_n via the explicit determination of the canonical basis of $V(\Lambda)$.

3.3.1 Ariki's theorem

Recall that we have denoted in Section 3.2.2 $V(\mathbf{s})$ for the $\mathcal{U}_q'(\widehat{\mathfrak{sl}_e})$-module $\mathcal{U}_q'(\widehat{\mathfrak{sl}_e})|\boldsymbol{\emptyset}, \mathbf{s}\rangle$, for $l \in \mathbb{Z}_{>0}$ and $\mathbf{s} \in \mathbb{Z}^l$. It gives a realisation of the abstract irreducible highest weight $\mathcal{U}_q'(\widehat{\mathfrak{sl}_e})$-module $V(\Lambda_\mathbf{s})$, where $\Lambda_\mathbf{s} = \mathrm{wt}(|\boldsymbol{\emptyset}, \mathbf{s}\rangle) \in P^+$. Its crystal graph has been described in Theorem 3.2.10 using Uglov l-partitions, and we have denoted

$$\mathcal{G}(\mathbf{s}) = \{G(\boldsymbol{\mu}, \mathbf{s}) \mid \boldsymbol{\mu} \in \Phi_\mathbf{s}\}$$

its canonical basis in the sense of Theorem 3.1.19.

For $\boldsymbol{\mu} \in \Phi_\mathbf{s}(n)$, write the decomposition of $G(\boldsymbol{\mu}, \mathbf{s})$ on the standard basis of l-partitions

$$G(\boldsymbol{\mu}, \mathbf{s}) = \sum_{\boldsymbol{\lambda} \vdash_l n} g_{\boldsymbol{\lambda},\boldsymbol{\mu}}(q)|\boldsymbol{\lambda}, \mathbf{s}\rangle.$$

Let $\mathcal{G}_1(\mathbf{s})$ be the specialisation at $q=1$ of $\mathcal{G}(\mathbf{s})$, that is define

$$G_1(\boldsymbol{\mu},\mathbf{s}) = \sum_{\boldsymbol{\lambda}\vdash_l n} g_{\boldsymbol{\lambda},\boldsymbol{\mu}}(1)|\boldsymbol{\lambda},\mathbf{s}\rangle \quad \text{for } \boldsymbol{\mu}\in\Phi_{\mathbf{s}}(n),$$

and

$$\mathcal{G}_1(\mathbf{s}) = \{G_1(\boldsymbol{\mu},\mathbf{s});\ \boldsymbol{\mu}\in\Phi_{\mathbf{s}}\}.$$

For $n\in\mathbb{Z}_{\geq 0}$, consider now the non-semisimple Ariki-Koike algebra $\mathbf{H}_{k,n}=\mathbf{H}_{k,n}^{(e,\mathbf{s})}$ associated to the specialisation

$$\begin{array}{rccc} \theta: & A & \longrightarrow & k \\ & u & \longmapsto & \eta \\ & V_i & \longmapsto & \eta^{s_i}, \end{array}$$

where η is a primitive root of unity of order e. Recall from Section 2.2 that we have denoted $d_{\boldsymbol{\lambda},M}$ the corresponding decomposition numbers, where $\boldsymbol{\lambda}$ runs over $\Pi_l(n)$ and M runs over $\mathrm{Irr}(\mathbf{H}_{k,n})$.

For $\boldsymbol{\lambda}\vdash_l n$ and $M\in\mathrm{Irr}(\mathbf{H}_{k,n})$, one can then define the following elements of the Fock space $\mathcal{F}_{\mathbf{s}}$

$$F(M,\mathbf{s}) = \sum_{\boldsymbol{\lambda}\vdash_l n} d_{\boldsymbol{\lambda},M}|\boldsymbol{\lambda},\mathbf{s}\rangle.$$

Then if we set $F(M,\mathbf{s}) = |\boldsymbol{\emptyset},\mathbf{s}\rangle$ for $n=0$, we have the following key result, now known as Ariki's theorem, proved in [2].

Theorem 3.3.1. *Suppose that* $\mathrm{char}(k)=0$. *Then*

$$\mathcal{G}_1(\mathbf{s}) = \{F(M,\mathbf{s})\ ;\ M\in\mathrm{Irr}(\mathbf{H}_{k,n}), n\in\mathbb{Z}_{\geq 0}\}.$$

In other words, if $\mathrm{char}(k)=0$, it is sufficient to compute the canonical basis of the irreducible highest weight $\mathcal{U}_q'(\widehat{\mathfrak{sl}_e})$-module $V(\mathbf{s})$ in order to recover the decomposition matrix D of $\mathbf{H}_{k,n}$.

3.3.2 Crystal structure for Cherednik algebras

Recent developments on cyclotomic rational Cherednik algebras have displayed even deeper connections between the theory of quantum groups and the representation theory of W_n. In fact, inspired by Ariki's work on Hecke algebras for W_n, Shan has developed a theory of crystals for Cherednik algebras similar to that of $\mathcal{F}_{\mathbf{s}}$.

Recall that we have defined in Section 2.3 i-induction and i-restriction functors $F_i(n)$ and $E_i(n)$ inside $\mathcal{O}_c=\bigoplus_{n\in\mathbb{Z}_{\geq 0}}\mathcal{O}_{c,n}$, the categories $\mathcal{O}$ for cyclotomic rational Cherednik algebras. For a module $M\in\mathcal{O}_c$, denote $\mathrm{head}(M)$ the *head* of M, i.e. the largest semisimple quotient of M (cf Section 2.3), and denote $\mathrm{soc}(M)$ the *socle* of M, i.e. the largest semisimple submodule of M.

Consider the basis $\mathscr{B}=\{[L(\boldsymbol{\lambda})]\ ;\ \boldsymbol{\lambda}\vdash_l n, n\in\mathbb{Z}_{\geq 0}\}$ of $[\mathcal{O}_n]$ (cf. Paragraph 2.3.2). Then, by [124, Proposition 6.2], we are ensured that

Lemma 3.3.2. *Equipped with the maps*

$$\hat{e}_i : [L(\boldsymbol{\lambda})] \mapsto [\mathrm{head}(E_i(L(\boldsymbol{\lambda})))] \quad \textit{and} \quad \hat{f}_i : [L(\boldsymbol{\lambda})] \mapsto [\mathrm{soc}(F_i(L(\boldsymbol{\lambda})))],$$

$\mathscr{B}$ is a perfect basis of $[\mathcal{O}_n]$.

Now, one can then define the coloured oriented graph $B(\mathcal{O}_c)$ as follows:

- vertices: all l-partitions $\boldsymbol{\lambda} \vdash_l n$ for $n \in \mathbb{Z}_{\geq 0}$.
- arrows: $[\boldsymbol{\mu}] \xrightarrow{i} [\boldsymbol{\lambda}]$ if and only if $L(\boldsymbol{\lambda}) = \mathrm{soc}(F_i(L(\boldsymbol{\mu})))$.

The following result is due to Shan [124, Theorem 6.3 (1)].

Proposition 3.3.3. *The graph $B(\mathcal{O}_c)$ encodes a crystal structure on $[\mathcal{O}_c]$.*

Recall that the parameter c is defined thanks to parameters $e \in \mathbb{Z}_{>1}$ and $\mathbf{s} \in \mathbb{Z}^l$ via the formulas (2.4). Consider now the Fock space $\mathcal{F}_{\mathbf{s}}$, seen as a $\mathcal{U}'_q(\widehat{\mathfrak{sl}_e})$-module. Clearly, $[\mathcal{O}_c]$ and $\mathcal{F}_{\mathbf{s}}$ are isomorphic as vector spaces, via the parametrisation of the bases of $[\mathcal{O}_c]$ by l-partitions. In fact, they have much more interesting common structures, as illustrates the following theorem ([124, Theorem 6.3]).

Theorem 3.3.4. *We have*

$$B(\mathcal{O}_c) \simeq B(\mathcal{F}_{\mathbf{s}}).$$

Remark 3.3.5. By $\simeq$, we mean isomorphic as coloured oriented graphs.

To prove this theorem, we need the following lemma. If B is a perfect basis of an integrable $\mathcal{U}(\widehat{\mathfrak{sl}_e})$-module V (see Definition 3.2.18), denote $B^+ = \{b \in B \mid e_i.b = 0\}$, and $V^+ = \mathrm{Span}(B^+)$. Then [124, Lemma 6.1] claims that

Lemma 3.3.6. *B^+ is a basis of V^+.*

Proof. For $v \in V^+$, write $v = \sum_{k=1}^r \alpha_k b_k$ where $r \in \mathbb{Z}_{>0}$, $\alpha_k \in \mathbb{C}^\times$ and $b_k \in B$ (all distinct). For all $i \in [\![0, e-1]\!]$, set $l_i = \max\{l_i(b_k) \ ; \ k \in [\![1, r]\!]\}$, and $K = \{k \in [\![1, r]\!] \mid l_i(b_k) = l_i\}$. Then we have

$$\begin{aligned}
0 &= e_i.v \quad \text{because } v \in V^+ \\
&= \sum_{k=1}^r \alpha_k e_i.b_k \\
&= \sum_{k=1}^r \alpha_k(\beta_k \hat{e}_i(b_k) + w_k) \quad \text{for some } \beta_k \in \mathbb{C}^\times, w_k \in V^{<l_i-1} \\
&= \sum_{k\in K} \alpha_k\beta_k\hat{e}_i(b_k) + \sum_{k\notin K} \alpha_k\beta_k\hat{e}_i(b_k) + \sum_{k=1}^r \beta_k w_k.
\end{aligned}$$

Now, because $w_k \in V^{<l_i-1}$ for all k, and by definition of K, we have

$$w = \sum_{k\notin K} \alpha_k\beta_k\hat{e}_i(b_k) + \sum_{k=1}^r \beta_k w_k \in V^{<l_i-1}.$$

Because the b_k are distinct, the $\hat{e}_i(b_k)$ are also distinct unless they are 0. Thus, since $l_i(\hat{e}_i(b_k)) = l_i - 1$, we necessarily have $\hat{e}_i(b_k) = 0$ for all $k \in K$, whence $l_i = 0$. Therefore, $\hat{e}_i(b_k) = 0$ for all $k \in [\![1, r]\!]$, i.e. belong to B^+, and B^+ is indeed a basis of V^+.

□

Proof of Theorem 3.3.4. One identifies the vector spaces $\mathcal{F}_{\mathbf{s}}$ and $[\mathcal{O}_c]$. By Proposition 3.2.19, $\mathscr{G} = \mathcal{G}_1^*(\mathcal{F}_{\mathbf{s}})$ is a perfect basis of the integrable $\mathcal{U}(\widehat{\mathfrak{sl}_e})$-module $\mathcal{F}_{\mathbf{s}}$ [2] and by Lemma 3.3.2, $\mathscr{B}$ is another perfect basis of $\mathcal{F}_{\mathbf{s}}$. Using Lemma 3.3.6, we know that $\mathscr{G}^+$ and $\mathscr{B}^+$ are two weight bases of $\mathcal{F}_{\mathbf{s}}^+$. Therefore, we have a bijection $\psi : \mathscr{G}^+ \longrightarrow \mathscr{B}^+$ verifying $\mathrm{wt}(\mathbf{G}_1^*(\boldsymbol{\lambda}, \mathbf{s})) = \mathrm{wt}(\psi(\mathbf{G}_1^*(\boldsymbol{\lambda}, \mathbf{s})))$, for all l-partition $\boldsymbol{\lambda}$. Because $\mathcal{F}_{\mathbf{s}}$ decomposes as a direct sum of highest weight irreducible modules, ψ extends to an automorphism of $\mathcal{F}_{\mathbf{s}}$. We conclude using [9, Theorem 5.37] which ensures that ψ induces an isomorphism of crystals.

□

The next natural problem is to try to make explicit this crystal structure, not only up to isomorphism. As a matter of fact, Losev has proved that it is exactly the crystal graph of Uglov's realisation of the Fock space (Theorem 3.2.10). His proof [107, Theorem 5.1] uses $\mathfrak{sl}_2$-categorification arguments (and requires the notion of *highest weight categorification*), with a systematic study of the weight spaces. We reformulate it as follows:

Theorem 3.3.7. *We have*

$$B(\mathcal{O}_c) = B(\mathcal{F}_{\mathbf{s}}).$$

Therefore, the considerations one can make on the crystal of the Fock space (for instance what is done in Chapter 5 of this thesis) have an interpretation in terms of representation theory of the associated cyclotomic rational Cherednik algebra, and conversely.

[2] Contrary to the original proof in [124], we use the dual canonical basis instead of the standard basis of all l-partitions, which is not perfect. Indeed, for instance, take $n = 3$, $l = 1$, $e = 3$, $\mathbf{s} = (0)$, and $\boldsymbol{\lambda} = (3.2^2.1)$. Then $\tilde{e}_1(\boldsymbol{\lambda}) = 0$ but $e_1.\boldsymbol{\lambda} \neq 0$, so that Point 2 of Definition 3.2.18 fails to hold.

Chapter 4

Canonical basic sets for Ariki-Koike algebras

Following Chapter 2.1, we consider a non-semisimple Ariki-Koike algebra $\mathbf{H}_{k,n} = \mathbf{H}_{k,n}^{(e,\mathbf{s})}$, with $e \in \mathbb{Z}_{>1} \sqcup \{\infty\}$ and $\mathbf{s} \in \mathbb{Z}^l$ for $l \in \mathbb{Z}_{>0}$. We have seen in Section 2.2.3 the definition of a canonical basic set for $\mathbf{H}_{k,n}$, see Definition 2.2.13. It formalises the fact that the decomposition matrix $D = D_{(e,\mathbf{s})}$ of $\mathbf{H}_{k,n}$ is upper unitriangular. Recall that it depends on another parameter $\mathbf{m} \in \mathbb{Q}^l$, called a weight sequence, which is used to define an order on the columns of D. In this chapter, we start by reviewing Geck and Jacon's results about the existence and construction of canonical basic sets as exposed in [53].

This motivates the first main result of this thesis, namely Theorem 4.5.2: we fully classify which values of the parameters yield a canonical basic set for $\mathbf{H}_{k,n}$ and which do not. These results can be found in the original paper [55].

In all this chapter, we will assume that $\mathrm{char}(k) = 0$, so that Ariki's theorem holds (Theorem 3.3.1).

Remark 4.0.8. In the rest of this chapter, we will consider $\mathcal{F}_\mathbf{s}$ as a $\mathcal{U}_q(\widehat{\mathfrak{sl}_e})$-module if $e < \infty$, and as a $\mathcal{U}_q(\mathfrak{sl}_\infty)$-module if $e = \infty$, cf. Remark 3.2.3.

4.1 Dependence on the multicharge

Recall that the algebra $\mathbf{H}_{k,n}$ is defined via the specialisation

$$\begin{array}{rccc} \theta_{(e,\mathbf{s})}: & A & \longrightarrow & k \\ & u & \longmapsto & \eta \\ & V_i & \longmapsto & \eta^{s_i}, \end{array}$$

where η has multiplicative order e, and $s_i \in \mathbb{Z}$ for all $i \in [\![1, l]\!]$.

Following the notations of Section 1.3.2, denote simply $\mathcal{C} = \mathcal{C}(\mathbf{s})$ and $\mathcal{C}_e = \mathcal{C}_e(\mathbf{s})$. Hence, every $\mathbf{r} \in \mathcal{C}$ writes $\mathbf{r} = w.\mathbf{s}$ for $w \in \widehat{\mathfrak{S}_l}$, and every $\mathbf{r} \in \mathcal{C}_e$ writes $\mathbf{r} = w'.\mathbf{s}$ for $w' \in \langle y_1, \dots, y_l \rangle$.

By definition of $\widehat{\mathfrak{S}_l}$ and of $\mathbf{H}_{k,n}^{(e,\mathbf{r})}$ (cf. Remark 2.2.6), one sees that for all $\mathbf{r} \in \mathcal{C}$, the specialised algebra $\mathbf{H}_{k,n}^{(e,\mathbf{r})}$ verifies the same defining relations as $\mathbf{H}_{k,n}^{(e,\mathbf{s})}$, whence

$$\mathbf{H}_{k,n}^{(e,\mathbf{r})} = \mathbf{H}_{k,n}^{(e,\mathbf{s})}. \tag{4.1}$$

Therefore, in this case, one can simply denote $\mathbf{H}_{k,n}$ regardless of the multicharge we choose. This also implies we can recover the matrix $D = D_{(e,\mathbf{s})}$ from any specialisation $\mathbf{H}_{k,n}^{(e,\mathbf{r})}$ where $\mathbf{s} \in \mathcal{C}$. However, Relation (4.1) does not say everything. It is important to understand the consequences of choosing another multicharge to get the decomposition matrix. In fact,

- If $\mathbf{r} \in \mathcal{C}_e$, then the decomposition maps are the same. Therefore, the decomposition matrices are strictly equal.
- If $\mathbf{s} \in \mathcal{C}$, then the decomposition maps do not necessarily coincide. In fact, if we have $\mathbf{s} = \sigma(\mathbf{r})$, for some $\sigma \in \mathfrak{S}_l$, denote $\theta := \theta_{(e,\mathbf{s})}$ and $\theta_\sigma := \theta_{(e,\mathbf{r})}$. Then we can write

  $$d_\theta([E^{\boldsymbol{\lambda}}]) = \sum_{M \in \mathrm{Irr}(\mathbf{H}_{k,n})} d_{\boldsymbol{\lambda},M}[M] \quad \text{and} \quad d_{\theta_\sigma}([E^{\boldsymbol{\lambda}}]) = \sum_{M \in \mathrm{Irr}(\mathbf{H}_{k,n})} d^\sigma_{\boldsymbol{\lambda},M}[M].$$

  Now since $\theta_\sigma(V_i) = \eta^{r_i} = \eta^{s_{\sigma(i)}} = \theta(V_{\sigma(i)})$, we have

  $$d^\sigma_{\boldsymbol{\lambda},M} = d_{\boldsymbol{\lambda}^\sigma,M}, \quad \forall \boldsymbol{\lambda} \vdash_l n\,, \forall M \in \mathrm{Irr}(\mathbf{H}_{k,n}) \tag{4.2}$$

  where $\boldsymbol{\lambda}^\sigma = (\lambda^{\sigma(1)}, \dots \lambda^{\sigma(l)})$.

  This means that the decomposition matrices $D_{(e,\mathbf{r})}$ and $D_{(e,\mathbf{s})}$ are equal up to a permutation of the rows. Equivalently, they are strictly equal (denoted by D) provided the parametrisation of the rows is changed: the row of D labeled by $\boldsymbol{\lambda}$ with respect to the parametrisation yielded by $\mathbf{s}$ is labeled by $\boldsymbol{\lambda}^\sigma$ with respect to the parametrisation yielded by $\mathbf{r} = \sigma(\mathbf{s})$.

To sum up, the specialised Ariki-Koike algebra only depends on $\mathcal{C}$. We denote it by $\mathbf{H}_{k,n}$. Hence we consider that for any $\mathbf{r} \in \mathcal{C}$, we obtain one genuine matrix, that we denote dy D, but that each element $\mathbf{r} \in \mathcal{C}$ yields a different (in general) parametrisation of the rows of D.

In our purpose to study canonical basic sets, it is crucial to understand which parametrisation we use. In fact, we will fix a multicharge $\mathbf{s}$ once and for all, and this will fix a parametrisation of the rows of D. Since the notion of canonical basic set depends on this parametrisation, see Definition 2.2.13, it depends on the multicharge. This justifies the introduction of the parameter $\mathbf{s}$ in Definition 2.2.13. As we have just noticed, the parametrisation of the rows of D is invariant in the class $\mathcal{C}_e$. Therefore, the following important property holds:

Proposition 4.1.1. *Let $\mathbf{r} \in \mathcal{C}_e$, and let $\mathbf{m} \in \mathbb{Q}^l$. If $(\mathbf{H}_{k,n}, \mathbf{s})$ admits a canonical basic set $\mathscr{B}$ with respect to $\ll_{\mathbf{m}}$, then $\mathscr{B}$ is the canonical basic set for $(\mathbf{H}_{k,n}, \mathbf{r})$ with respect to $\ll_{\mathbf{m}}$.*

However, this is not true for general $\mathbf{r} \in \mathcal{C}$. For such a multicharge, it is sometimes possible to find a canonical basic set for $(\mathbf{H}_{k,n}, \mathbf{r})$, even if $(\mathbf{H}_{k,n}, \mathbf{s})$ does not admit any canonical basic set and both algebras are equal, see Example 4.5.4. Also, if $\mathscr{B}$ is the canonical basic set for $(\mathbf{H}_{k,n}, \mathbf{s})$, it is sometimes possible to find $\mathbf{r} \in \mathcal{C}$ such that $(\mathbf{H}_{k,n}, \mathbf{r})$ admits a canonical basic set $\mathscr{B}'$ and $\mathscr{B}' \neq \mathscr{B}$, see Example 4.3.3.

Remark 4.1.2. Note that it is already known that canonical basic sets do not always exist. For instance, in level 2, that is, when $\mathbf{H}_{k,n}$ can be seen as an Iwahori-Hecke algebra of type B_n, Geck and Jacon have computed in [53, Example 3.1.15 (c)] a decomposition matrix, associated to a specialisation, with $k = \mathbb{F}_2(v)$ and $n = 2$, which does not admit any canonical basic set with respect to the **a**-function. Recall however that we have assumed that $\mathrm{char}(k) = 0$, so we will not have this kind of issues.

In the rest of this chapter, we will first recall some existence results of canonical basic sets for $(\mathbf{H}_{k,n}, \mathbf{s})$ with respect to $\ll_{\mathbf{m}}$, before fully classifying the values the parameter $\mathbf{m}$ (the weight sequence) which yield canonical basic sets for $(\mathbf{H}_{k,n}, \mathbf{s})$ with respect to $\ll_{\mathbf{m}}$.

Remark 4.1.3. The existence of canonical basic sets for Hecke algebras of more general reflection groups have been notably studied in [26], [27] and [28].

4.2 Existence of canonical basic sets for appropriate parameters

Consider the specialised Ariki-Koike algebra $\mathbf{H}_{k,n} = \mathbf{H}_{k,n}^{(e,\mathbf{s})}$ where $e \in \mathbb{Z}_{>1} \cup \{\infty\}$, and $\mathbf{s} = (s_1, \dots, s_l) \in \mathbb{Z}^l$. We want to find a canonical basic set for $(\mathbf{H}_{k,n}, \mathbf{s})$, in the sense of Definition 2.2.13. The following results prove that it is always possible to find $\mathbf{m} \in \mathbb{Q}^l$ such that $(\mathbf{H}_{k,n}, \mathbf{s})$ admits a canonical basic set with respect to $\ll_{\mathbf{m}}$. Besides, they can be explicitly described, either "directly" (FLOTW l-partitions) or recursively (Uglov l-partitions). These results can be found in [53].

4.2.1 FLOTW multipartitions as canonical basic sets

We have the following result by Geck and Jacon.

Theorem 4.2.1 ([53, Theorem 5.8.2])**.** *Let* $\mathbf{s} \in \mathscr{S}_e^l$, $\mathbf{v} = (v_1, \dots, v_l) \in \mathbb{Q}^l$ *such that* $i < j \Rightarrow 0 < v_j - v_i < e$, *and set* $\mathbf{m} = \mathbf{s} - \mathbf{v} = (s_1 - v_1, \dots, s_l - v_l)$. *Then* $(\mathbf{H}_{k,n}, \mathbf{s})$ *admits a canonical basic set with respect to* $\ll_{\mathbf{m}}$, *namely the set* $\Psi_{\mathbf{s}}(n)$ *of FLOTW l-partitions of rank n (see Definition 1.4.1).*

Remark 4.2.2. The definition of FLOTW multipartitions naturally extends for $e = \infty$. Indeed, we set $\mathscr{S}_\infty^l = \left\{\mathbf{s} \in \mathbb{Z}^l \mid 0 \leq s_c - s_{c'} \text{ for } c < c'\right\}$, and an l-partition $\boldsymbol{\lambda}$ is said to be FLOTW for $e = \infty$ if for all $1 \leq c \leq l-1$, $\lambda_a^c \geq \lambda_{a+s_{c+1}-s_c}^{c+1}$ for all $a \geq 1$. In other terms, a multipartition is FLOTW for $e = \infty$ if and only if its symbol is semistandard, cf. Definition 1.1.8.

4.2.2 Uglov multipartitions as canonical basic sets

We introduce one more notation. Let $\mathbf{r} = (r_1, \ldots, r_l) \in \mathcal{C}$. For $\mathbf{m} = (m_1, \ldots, m_l) \in \mathbb{Q}^l$, we set $\mathbf{v} = (v_1, \ldots, v_l) = (r_1 - m_1, \ldots, r_l - m_l)$, and we define

$$\mathcal{D}_{\mathbf{r}} = \Big\{ \mathbf{m} \in \mathbb{Q}^l \,|\, i < j \Rightarrow 0 < v_j - v_i < e \Big\}.$$

Using Uglov's canonical basis of the Fock space, Geck and Jacon have proved the following result about canonical basic sets:

Theorem 4.2.3 ([53, Theorem 6.7.2])**.** *Suppose that* $\mathrm{char}(k) = 0$. *Let* $\mathbf{s} \in \mathbb{Z}^l$ *and* $\mathbf{m} \in \mathbb{Q}^l$. *If* $\mathbf{m} \in \mathcal{D}_{\mathbf{s}}$, *then* $(\mathbf{H}_{k,n}, \mathbf{s})$ *admits a canonical basic set with respect to* $\ll_{\mathbf{m}}$, *namely the set* $\Phi_{\mathbf{s}}(n)$ *of Uglov* l*-partitions of rank* n *(see Corollary 3.2.12).*

Sketch of Proof. One can prove that the matrix of Uglov's canonical basis of $\mathcal{F}_{\mathbf{s}}$ is unitriangular with respect to $\ll_{\mathbf{m}}$, i.e. for all $\boldsymbol{\mu} \in \Phi_{\mathbf{r}}(n)$,

$$\mathbf{G}(\boldsymbol{\mu}, \mathbf{s}) = |\boldsymbol{\mu}, \mathbf{s}\rangle + \sum_{\substack{\boldsymbol{\lambda} \vdash_l n, \boldsymbol{\lambda} \neq \boldsymbol{\mu} \\ \boldsymbol{\mu} \ll_{\mathbf{m}} \boldsymbol{\lambda}}} g_{\boldsymbol{\lambda}, \boldsymbol{\mu}}(q) |\boldsymbol{\lambda}, \mathbf{s}\rangle,$$

where the notations are those of Paragraph 3.3.1. One concludes using Theorem 3.3.1 which claims that the matrix D is obtained by specialising at $q = 1$ the matrix of the canonical basis of $\mathcal{F}_{\mathbf{s}}$ and keeping only the rows labeled by the Uglov l-partitions.

□

We now want to solve a more general problem, namely we wish to review the existence or non-existence of a canonical basic set for $(\mathbf{H}_{k,n}, \mathbf{s})$ with respect to $\ll_{\mathbf{m}}$, depending on the values of $\mathbf{m}$, and explicitly describe these sets when they exist.

In the following sections, we will denote by $\mathscr{P}$ the following subset of $\mathbb{Q}^l$:

$$\mathscr{P} = \Big\{ m \in \mathbb{Q}^l \,|\, \exists\, i \neq j \text{ such that } (s_i - m_i) - (s_j - m_j) \in e\mathbb{Z} \Big\}.$$

Precisely, $\mathscr{P}$ consists in the union of the hyperplanes

$$\begin{aligned} \mathcal{P}_{i,j}(\mathbf{r}) &= \Big\{ \mathbf{m} \in \mathbb{Q}^l \,|\, (r_i - m_i) - (r_j - m_j) = 0 \Big\} \\ &= \Big\{ \mathbf{m} \in \mathbb{Q}^l \,|\, v_i - v_j = 0 \Big\} \quad (\text{where } v_i = r_i - m_i \ \forall i \in [\![1, l]\!]) \end{aligned}$$

over all $\mathbf{r} \in \mathcal{C}_e$ and all $1 \leq i < j \leq l$.

Indeed, for $k \in \mathbb{Z}$, we have

$$\begin{aligned} (s_i - m_i) - (s_j - m_j) = ke &\Leftrightarrow (s_i - m_i) - (s_j + ke - m_j) = 0 \\ &\Leftrightarrow (r_i - m_i) - (r_j - m_j) = 0 \\ &\Leftrightarrow v_i - v_j = 0 \end{aligned}$$

with $\mathbf{r} = (r_1, \dots, r_i, \dots, r_j, \dots, r_l) = (s_1, \dots, s_i, \dots, s_j + ke, \dots, s_l)$.

Clearly, this is not a disjoint union, since $\mathcal{P}_{i,j}(\mathbf{r}) = \mathcal{P}_{i,j}(\tilde{\mathbf{r}})$ whenever $\tilde{r}_i = r_i + pe$ and $\tilde{r}_j = r_j + pe$ for some $p \in \mathbb{Z}$. Also, when $l > 2$, the hyperplanes $\mathcal{P}_{i,j}(\mathbf{r})$ and $\mathcal{P}_{i',j'}(\mathbf{r})$ always intersect, even when $(i, j) \neq (i', j')$.

Now, for $\mathbf{m} \in \mathbb{Q}^l$, we can always find a multicharge $\mathbf{r} \in \mathcal{C}_e$ which is "close" to $\mathbf{m}$ in the following sense. In $\mathbb{Q}^l$, consider the closed balls $B_{e/2}(\mathbf{r})$, with respect to the infinity norm, of radius $e/2$ and centered at $\mathbf{r}$, for $\mathbf{r} \in \mathcal{C}_e$. By definition of $\mathcal{C}_e$, it is clear that

$$\bigcup_{\mathbf{r} \in \mathcal{C}_e} B_{e/2}(\mathbf{r}) = \mathbb{Q}^l, \quad \text{and} \quad \bigcap_{\mathbf{r} \in \mathcal{C}_e} B_{e/2}(\mathbf{r}) = \bigcup_{\mathbf{r} \in \mathcal{C}_e} \partial B_{e/2}(\mathbf{r}).$$

In other terms, these balls cover $\mathbb{Q}^l$, and only their boundaries intersect. Hence, there is a particular $\mathbf{r} = (r_1, \dots, r_l) \in \mathcal{C}_e$ such that $\mathbf{m} \in B_{e/2}(\mathbf{r})$, and this multicharge is unique if $\mathbf{m}$ is not on the boundary of the ball. If it belongs to the boundary, then this means that there exists $i \in [\![1, l]\!]$ such that $|v_i| = |r_i - m_i| = e/2$. In this case, we make $\mathbf{r}$ unique by setting $v_i = e/2$.

Definition 4.2.4. The element $\mathbf{r}$ thus obtained is called the $\mathbf{m}$-*adapted multicharge.*

The following easy lemma will be useful in the last two sections.

Lemma 4.2.5. *Let $\mathbf{m} \in \mathbb{Q}^l$. Suppose that $\mathbf{m} \in \mathcal{P}_{i,j}(\mathbf{r}')$ for some $i < j$ and some $\mathbf{r}' \in \mathcal{C}_e$. Then $\mathcal{P}_{i,j}(\mathbf{r}) = \mathcal{P}_{i,j}(\mathbf{r}')$, where $\mathbf{r}$ is the $\mathbf{m}$-adapted multicharge.*

Proof. The multicharge $\mathbf{r}$ verifies in particular $0 \leq |r_i - m_i| \leq e/2$ and $0 \leq |r_j - m_j| \leq e/2$. Also, because $\mathbf{r}', \mathbf{r} \in \mathcal{C}_e$, we can write $r_i = r'_i + pe$ for some $p \in \mathbb{Z}$. This gives

$$0 \leq |r'_i - m_i + pe| \leq e/2,$$

i.e.

$$0 \leq |r'_j - m_j + pe| \leq e/2 \text{ since } \mathbf{m} \in \mathcal{P}_{i,j}(\mathbf{r}').$$

Hence we have $r_j = r'_j + pe$, which implies that $\mathcal{P}_{i,j}(\mathbf{r}) = \mathcal{P}_{i,j}(\mathbf{r}')$. □

As a consequence, if $\mathbf{m} \in \mathscr{P}$, it writes a priori $\mathbf{m} \in \bigcap_{\substack{(i,j) \in J \\ \mathbf{r}' \in S}} \mathcal{P}_{i,j}(\mathbf{r}')$ for some index set J and some $S \subset \mathcal{C}_e$, but, a posteriori, we can simply write $\mathbf{m} \in \bigcap_{(i,j) \in J} \mathcal{P}_{i,j}(\mathbf{r})$, where $\mathbf{r}$ is the $\mathbf{m}$-adapted multicharge.

4.3 Canonical basic sets for regular weight sequences

If $\mathbf{m} \in \mathbb{Q}^l \backslash \mathscr{P}$, we say that $\mathbf{m}$ is *regular*. In this section, we show that any regular $\mathbf{m}$ defines an order $\ll_{\mathbf{m}}$ with respect to which $\mathbf{H}_{k,n}$ admits a canonical basic set. We use the fact that, for $\mathbf{r} \in \mathcal{C}_e$, if $\mathscr{B}$ is the canonical basic set for $(\mathbf{H}_{k,n}, \mathbf{r})$ with respect to $\ll_{\mathbf{m}}$, then $\mathscr{B}$ is the canonical basic set for $(\mathbf{H}_{k,n}, \mathbf{s})$ with respect to $\ll_{\mathbf{m}}$ (see Proposition 4.1.1). We first study the case $l = 2$, and then the general case.

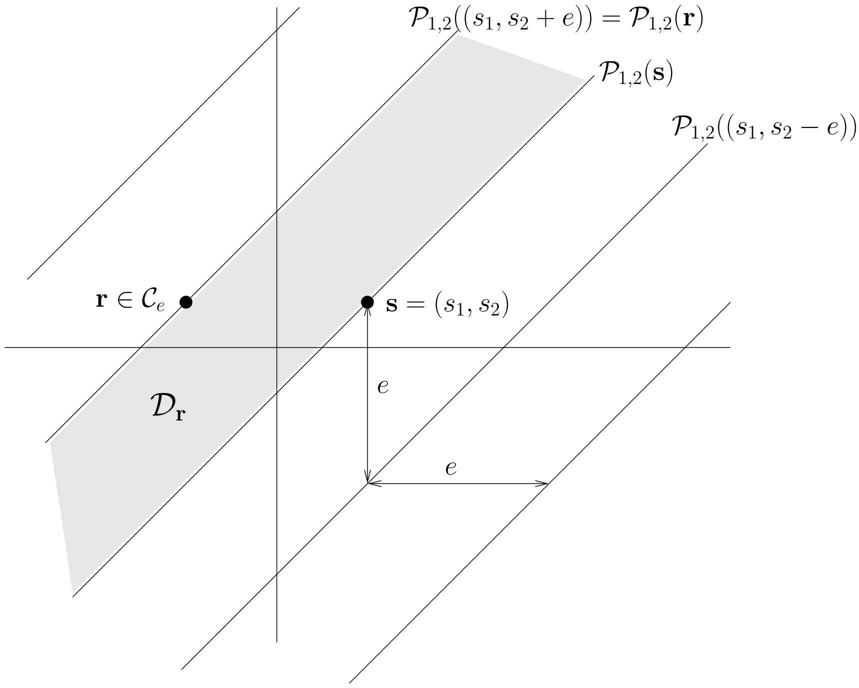


Figure 4.1: The set $\mathscr{P}$ and the domains $\mathcal{D}_\mathbf{r}$ in level 2.

4.3.1 Level 2 case

Here we have $\mathbf{s} = (s_1, s_2) \in \mathbb{Z}^2$, and $\mathscr{P}$ is just a collection of parallel lines, namely the lines passing through $(r_1, r_2) \in \mathcal{C}_e$ with slope 1. The set $\mathcal{D}_\mathbf{r}$ is the domain strictly between the lines passing through (r_1, r_2) and $(r_1, r_2 - e)$, see Figure 4.1.

Notation: For $\mathbf{r} = (r_1, r_2) \in \mathcal{C}$, we denote $\tilde{\mathbf{r}} = (r_1, r_2 + e)$.

Proposition 4.3.1. *Let $\mathbf{m} = (m_1, m_2) \in \mathbb{Q}^2 \backslash \mathscr{P}$. Then $(\mathbf{H}_{k,n}, \mathbf{s})$ admits a canonical basic set with respect to $\ll_\mathbf{m}$, namely either $\Phi_\mathbf{r}(n)$ or $\Phi_{\tilde{\mathbf{r}}}(n)$, where $\mathbf{r} \in \mathcal{C}_e$ is explicitly determined.*

Proof. The idea is to show that any such $\mathbf{m}$ belongs to a certain $\mathcal{D}_{\hat{\mathbf{r}}}$, for $\hat{\mathbf{r}} \in \mathcal{C}_e$.

Consider the $\mathbf{m}$-adapted multicharge $\mathbf{r}$ (cf. Definition 4.2.4). It verifies, in particular, $0 \leq |r_i - m_i| \leq \frac{e}{2}, i = 1, 2.$. We set, as usual, $v_i = r_i - m_i$, so that we have $0 \leq |v_1 - v_2| \leq e$. The fact that $\mathbf{m}$ is not located on a hyperplane of $\mathscr{P}$ (and that $\mathbf{r}$ is obtained from $\mathbf{r}$ after translation of each coordinate by an element of $e\mathbb{Z}$) ensures that one can never have $v_1 = v_2$. Thus $0 < |v_1 - v_2| < e$. Now,

- If $0 < v_2 - v_1 < e$, then $\mathbf{m} \in \mathcal{D}_\mathbf{r}$. By Theorem 4.2.3, the set $\Phi_\mathbf{r}(n)$ is the canonical basic set for the algebra $\mathbf{H}_{k,n}^{(e,\mathbf{r})}$ with respect to the order $\ll_\mathbf{m}$. Therefore, by Proposition 4.1.1, $\Phi_\mathbf{r}(n)$ is the canonical basic set for $(\mathbf{H}_{k,n}, \mathbf{s})$ with respect to $\ll_\mathbf{m}$.
- If $0 < v_1 - v_2 < e$, then $0 < (v_2 + e) - v_1 < e$. Hence $\mathbf{m} \in \mathcal{D}_{\tilde{\mathbf{r}}}$ (where we recall that $\tilde{\mathbf{r}} = (s_1, s_2 + e)$), so that by Theorem 4.2.3, the set $\Phi_{\tilde{\mathbf{r}}}(n)$ is the canonical basic set

for $(\mathbf{H}_{k,n}, \tilde{\mathbf{r}})$ with respect to the order $\ll_{\mathbf{m}}$. Since $\tilde{\mathbf{r}} \in \mathcal{C}_e$, using Proposition 4.1.1, $\Phi_{\tilde{\mathbf{r}}}(n)$ is the canonical basic set for $(\mathbf{H}_{k,n}, \mathbf{s})$ with respect to $\ll_{\mathbf{m}}$.

□

Remark 4.3.2. In level 2, the domains $\mathcal{D}_{\mathbf{r}}$, $\mathbf{r} \in \mathcal{C}_e$, actually tile $\mathbb{Q}^2 \backslash \mathscr{P}$. In higher level this does not hold anymore, and we need to find other canonical basic sets than Uglov multipartitions.

Note that we can sometimes find different canonical basic sets for $(\mathbf{H}_{k,n}, \mathbf{s})$ and $(\mathbf{H}_{k,n}, \mathbf{r})$ with respect to the same order $\ll_{\mathbf{m}}$ if $\mathbf{r} \in \mathcal{C} \backslash \mathcal{C}_e$, as mentioned in Section 4.1. This is what the following example shows.

Example 4.3.3. Let $l = 2$, $e = 4$, $\mathbf{s} = (1, 0)$ and $\mathbf{r} = \mathbf{s}^{\sigma} = (0, 1)$ (where $\sigma = (12)$). Then $\mathbf{r} \notin \mathcal{C}_e$. Take $\mathbf{m} = (1, -1)$. Then $\mathbf{m} \in \mathcal{D}_{\mathbf{s}}$ since $\mathbf{s} - \mathbf{m} = (0, 1)$. Hence $\Phi_{\mathbf{s}}(n)$ is the canonical basic set for $(\mathbf{H}_{k,n}, \mathbf{s})$. Besides, $\mathbf{m} \in \mathcal{D}_{\mathbf{r}}$ since $\mathbf{r} - \mathbf{m} = (-1, 2)$. Hence $\Phi_{\mathbf{r}}(n)$ is the canonical basic set for $(\mathbf{H}_{k,n}, \mathbf{r})$, but $\Phi_{\mathbf{r}}(n) \neq \Phi_{\mathbf{s}}(n)$ for $n \geq 2$, which one can easily check, see Example B.0.10.

4.3.2 Higher level case

Throughout this thesis, we will use the following notations. For $\boldsymbol{\alpha} = (\alpha_1, \dots \alpha_l) \in \mathbb{Q}^l$ and $\sigma \in \mathfrak{S}_l$, we denote $\boldsymbol{\alpha}^{\sigma} = (\alpha_{\sigma(1)}, \dots \alpha_{\sigma(l)})$. Similarly, for $\boldsymbol{\lambda} = (\lambda^1, ..., \lambda^l) \vdash_l n$, we write $\boldsymbol{\lambda}^{\sigma} = (\lambda^{\sigma(1)}, \dots, \lambda^{\sigma(l)})$.

Let $\mathbf{m} = (m_1, ..., m_l)$ be an element of $\mathbb{Q}^l \backslash \mathscr{P}$.

Proposition 4.3.4. *Let $\mathbf{r} \in \mathcal{C}$ and $\sigma \in \mathfrak{S}_l$. If $\Phi_{\mathbf{r}}(n)$ is the canonical basic set for $(\mathbf{H}_{k,n}, \mathbf{r})$ with respect to $\ll_{\mathbf{m}}$, then the set*

$$\sigma(\Phi_{\mathbf{r}}(n)) := \{\boldsymbol{\lambda}^{\sigma}\ ;\ \boldsymbol{\lambda} \in \Phi_{\mathbf{r}}(n)\}$$

of σ-twisted Uglov l-partitions is the canonical basic set for $(\mathbf{H}_{k,n}, \mathbf{r}^{\sigma})$ with respect to $\ll_{\mathbf{m}^{\sigma}}$.

Proof. In order to prove this result, we need to define a *twisted Fock space* $\mathcal{F}^{\sigma}_{\mathbf{r}^{\sigma}}$, which, as a vector space, is the Fock space $\mathcal{F}_{\mathbf{r}^{\sigma}}$, but has a σ-twisted $\mathcal{U}_q(\widehat{\mathfrak{sl}_e})$-action.

Recall that the action of $\mathcal{U}_q(\widehat{\mathfrak{sl}_e})$ on the Fock space $\mathcal{F}_{\mathbf{r}}$ is derived from an order on the i-nodes of l-partitions. We define a twisted order on the removable and addable i-nodes of a multipartition in the following way: let $\gamma = (a, b, c)$ and $\gamma' = (a', b', c')$ be two removable or addable i-nodes of $\boldsymbol{\lambda} \vdash_l n$. We write

$$\gamma \prec^{\sigma}_{\mathbf{r}^{\sigma}} \gamma' \text{ if } \begin{cases} b - a + r_{\sigma(c)} < b' - a' + r_{\sigma(c')} \text{ or} \\ b - a + r_{\sigma(c)} = b' - a' + r_{\sigma(c')} \quad \text{and} \quad \sigma(c) > \sigma(c'). \end{cases}$$

Now, if $\gamma = (a, b, c)$ is a removable (resp. addable) i-node of $\boldsymbol{\lambda} = (\lambda^1, ..., \lambda^l)$, then $\gamma^\sigma := (a, b, \sigma^{-1}(c))$ is a removable (resp. addable) i-node of $\boldsymbol{\lambda}^\sigma := (\lambda^{\sigma(1)}, ..., \lambda^{\sigma(l)})$, so that we have

$$\gamma^\sigma \prec^\sigma_{\mathbf{r}^\sigma} \gamma'^\sigma \Leftrightarrow \gamma \prec_{\mathbf{r}} \gamma'.$$

This order enables us to define the numbers $N_i^{\prec^\sigma}(\boldsymbol{\lambda}, \boldsymbol{\mu})$ and $N_i^{\succ^\sigma}(\boldsymbol{\lambda}, \boldsymbol{\mu})$. Let $\boldsymbol{\lambda} \vdash_l n$ and $\boldsymbol{\mu} \vdash_l n+1$ such that $[\boldsymbol{\mu}] = [\boldsymbol{\lambda}] \cup \{\gamma\}$ where γ is an i-node. Then set

$$\begin{aligned} N_i^{\prec^\sigma}(\boldsymbol{\lambda}, \boldsymbol{\mu}) = & \sharp\{\text{addable } i\text{-nodes } \gamma' \text{ of } \boldsymbol{\lambda} \text{ such that } \gamma' \prec^\sigma_{\mathbf{r}^\sigma} \gamma\} - \\ & \sharp\{\text{removable } i\text{-nodes } \gamma' \text{ of } \boldsymbol{\mu} \text{ such that } \gamma' \prec^\sigma_{\mathbf{r}^\sigma} \gamma\} \end{aligned}$$

and

$$\begin{aligned} N_i^{\succ^\sigma}(\boldsymbol{\lambda}, \boldsymbol{\mu}) = & \sharp\{\text{addable } i\text{-nodes } \gamma' \text{ of } \boldsymbol{\lambda} \text{ such that } \gamma' \succ^\sigma_{\mathbf{r}^\sigma} \gamma\} - \\ & \sharp\{\text{removable } i\text{-nodes } \gamma' \text{ of } \boldsymbol{\mu} \text{ such that } \gamma' \succ^\sigma_{\mathbf{r}^\sigma} \gamma\} \end{aligned}$$

We abuse the notation by denoting σ the isomorphism of vector spaces

$$\sigma : \begin{array}{ccc} \mathcal{F}_{\mathbf{r}} & \longrightarrow & \mathcal{F}_{\mathbf{r}^\sigma} \\ |\boldsymbol{\lambda}, \mathbf{r}\rangle & \longmapsto & |\boldsymbol{\lambda}^\sigma, \mathbf{r}^\sigma\rangle \end{array}$$

Now we want do define a twisted action of $\mathcal{U}_q(\widehat{\mathfrak{sl}_e})$ on $\mathcal{F}_{\mathbf{r}^\sigma}$.

The action of e_i and f_i, denoted by $e_i^\sigma.\ |\boldsymbol{\lambda}^\sigma, \mathbf{r}^\sigma\rangle$ and $f_i^\sigma.\ |\boldsymbol{\lambda}^\sigma, \mathbf{r}^\sigma\rangle$, are defined as follows:

$$e_i^\sigma.\ |\boldsymbol{\lambda}^\sigma, \mathbf{r}^\sigma\rangle = \sum_{\mathrm{res}_e([\boldsymbol{\lambda}^\sigma] \setminus [\boldsymbol{\mu}^\sigma]) = i} q^{-N_i^{\prec^\sigma}(\boldsymbol{\mu}^\sigma, \boldsymbol{\lambda}^\sigma)}\ |\boldsymbol{\mu}^\sigma, \mathbf{r}^\sigma\rangle.$$

Then we have

$$\begin{aligned} e_i^\sigma.\ |\boldsymbol{\lambda}^\sigma, \mathbf{r}^\sigma\rangle &= \sum_{\mathrm{res}_e([\boldsymbol{\lambda}] \setminus [\boldsymbol{\mu}]) = i} q^{-N_i^{\prec}(\boldsymbol{\mu}, \boldsymbol{\lambda})} \sigma(\,|\boldsymbol{\mu}, \mathbf{r}\rangle) \\ &= \sigma\Big(\sum_{\mathrm{res}_e([\boldsymbol{\lambda}] \setminus [\boldsymbol{\mu}]) = i} q^{-N_i^{\prec}(\boldsymbol{\mu}, \boldsymbol{\lambda})}\ |\boldsymbol{\mu}, \mathbf{r}\rangle\Big) \end{aligned}$$

that is, e_i^σ acts as $\sigma e_i \sigma^{-1}$.

Similarly, if we set

$$f_i^\sigma.\ |\boldsymbol{\lambda}^\sigma, \mathbf{r}^\sigma\rangle = \sum_{\mathrm{res}_e([\boldsymbol{\mu}^\sigma] \setminus [\boldsymbol{\lambda}^\sigma]) = i} q^{-N_i^{\succ^\sigma}(\boldsymbol{\lambda}^\sigma, \boldsymbol{\mu}^\sigma)}\ |\boldsymbol{\mu}^\sigma, \mathbf{r}^\sigma\rangle,$$

we have

$$\begin{aligned} f_i^\sigma.\ |\boldsymbol{\lambda}^\sigma, \mathbf{r}^\sigma\rangle &= \sum_{\mathrm{res}_e([\boldsymbol{\mu}] \setminus [\boldsymbol{\lambda}]) = i} q^{-N_i^{\succ}(\boldsymbol{\lambda}, \boldsymbol{\mu})} \sigma(\,|\boldsymbol{\mu}, \mathbf{r}\rangle) \\ &= \sigma\Big(\sum_{\mathrm{res}_e([\boldsymbol{\mu}] \setminus [\boldsymbol{\lambda}]) = i} q^{-N_i^{\prec}(\boldsymbol{\lambda}, \boldsymbol{\mu})}\ |\boldsymbol{\mu}, \mathbf{r}\rangle\Big) \end{aligned}$$

that is, f_i^σ acts as $\sigma f_i \sigma^{-1}$.

Hence by Theorem 3.2.2, these new formulas, combined with the formulas

$$t_i.\ |\boldsymbol{\lambda}^\sigma, \mathbf{r}^\sigma\rangle = q^{N_i(\boldsymbol{\lambda})} |\boldsymbol{\lambda}^\sigma, \mathbf{r}^\sigma\rangle$$

and

$$\mathfrak{d}.\, |\boldsymbol{\lambda}^{\sigma}, \mathbf{r}^{\sigma}\rangle = -(\Delta(\mathbf{r}, n) + N_{\mathfrak{d}}(\boldsymbol{\lambda}))|\boldsymbol{\lambda}^{\sigma}, \mathbf{r}^{\sigma}\rangle$$

endow $\mathcal{F}_{\mathbf{r}^{\sigma}}$ with the structure of an integrable $\mathcal{U}_q(\widehat{\mathfrak{sl}_e})$-module, that we denote by $\mathcal{F}^{\sigma}_{\mathbf{r}^{\sigma}}$.

According to what we have done in Chapter 3, for $e < \infty$, we will consider from now on only the $\mathcal{U}'_q(\widehat{\mathfrak{sl}_e})$-module (respectively $\mathcal{U}_q(\mathfrak{sl}_{\infty})$-module) structure on $\mathcal{F}_{\mathbf{r}^{\sigma}}$ by forgetting the action of the generator $\mathfrak{d}$ (respectively, by forgetting $\mathfrak{d}$ and replacing residues by contents), see Remark 3.2.3.

We continue the construction as in the non-twisted case. Denote by $V(\mathbf{r}^{\sigma})^{\sigma}$ the $\mathcal{U}'_q(\widehat{\mathfrak{sl}_e})$-submodule of $\mathcal{F}^{\sigma}_{\mathbf{r}^{\sigma}}$ generated by the empty l-partition $|\boldsymbol{\emptyset}, \mathbf{r}^{\sigma}\rangle$. This is an irreducible highest weight $\mathcal{U}'_q(\widehat{\mathfrak{sl}_e})$-module for this twisted action, and the crystal basis of $V(\mathbf{r})$ is mapped to the one of $V(\mathbf{r}^{\sigma})^{\sigma}$ by the isomorphism σ. In particular the vertices of the crystal graph of $V(\mathbf{r}^{\sigma})^{\sigma}$ are the *σ-twisted Uglov l-partitions*:

$$\sigma(\Phi_{\mathbf{r}}(n)) := \{(\lambda^{\sigma(1)}, ..., \lambda^{\sigma(l)})\, ;\, (\lambda^1, ..., \lambda^l) \in \Phi_{\mathbf{r}}(n)\}.$$

It is an indexing set for the canonical basis of $V(\mathbf{r}^{\sigma})^{\sigma}$. Now this basis is also obtained from the canonical basis of $V(\mathbf{r})$ by applying σ. That is, if we denote by $\mathbf{G}^{\sigma}(\boldsymbol{\lambda}, \mathbf{r}^{\sigma})$ ($\boldsymbol{\lambda} \in \sigma(\Phi_{\mathbf{r}}(n))$) the elements of the canonical basis of $V(\mathbf{r}^{\sigma})^{\sigma}$, we have:

$$\sigma(\mathbf{G}(\boldsymbol{\lambda}, \mathbf{r})) = \mathbf{G}^{\sigma}(\boldsymbol{\lambda}^{\sigma}, \mathbf{r}^{\sigma}). \tag{4.3}$$

Write $\mathbf{G}^{\sigma}(\boldsymbol{\lambda}, \mathbf{r}^{\sigma}) = \sum_{\boldsymbol{\mu}^{\sigma} \vdash_l n} g_{\boldsymbol{\mu}^{\sigma}, \boldsymbol{\lambda}^{\sigma}}(q)\, |\boldsymbol{\mu}^{\sigma}, \mathbf{r}^{\sigma}\rangle$ the decomposition of $\mathbf{G}^{\sigma}(\boldsymbol{\lambda}, \mathbf{r}^{\sigma})$ on the basis of all l-partitions. By (4.3), we have

$$\begin{aligned} \sigma(\sum_{\boldsymbol{\mu} \vdash_l n} g_{\boldsymbol{\mu}, \boldsymbol{\lambda}}(q)\, |\boldsymbol{\mu}, \mathbf{r}\rangle) &= \sum_{\boldsymbol{\mu}^{\sigma} \vdash_l n} g_{\boldsymbol{\mu}^{\sigma}, \boldsymbol{\lambda}^{\sigma}}(q)\, |\boldsymbol{\mu}^{\sigma}, \mathbf{r}^{\sigma}\rangle \\ \text{i.e. } \sum_{\boldsymbol{\mu} \vdash_l n} g_{\boldsymbol{\mu}, \boldsymbol{\lambda}}(q)\, |\boldsymbol{\mu}^{\sigma}, \mathbf{r}^{\sigma}\rangle &= \sum_{\boldsymbol{\mu} \vdash_l n} g_{\boldsymbol{\mu}^{\sigma}, \boldsymbol{\lambda}^{\sigma}}(q)\, |\boldsymbol{\mu}^{\sigma}, \mathbf{r}^{\sigma}\rangle\, . \end{aligned}$$

Hence $\forall \boldsymbol{\lambda} \in \Phi_{\mathbf{r}}(n)$, $\forall \boldsymbol{\mu} \vdash_l n$, we have

$$g_{\boldsymbol{\mu}^{\sigma}, \boldsymbol{\lambda}^{\sigma}}(q) = g_{\boldsymbol{\mu}, \boldsymbol{\lambda}}(q). \tag{4.4}$$

In particular, this is true at $q = 1$.

By Ariki's theorem (Theorem 3.3.1), which holds for any realisation of the highest weight $\mathcal{U}'_q(\widehat{\mathfrak{sl}_e})$-module $V(\mathbf{s})$, the matrix $(g_{\boldsymbol{\mu}, \boldsymbol{\lambda}}(1))_{\boldsymbol{\mu} \vdash_l n, \boldsymbol{\lambda} \in \Phi_{\mathbf{r}}(n)}$ is the decomposition matrix D of $\mathbf{H}_{k,n}$. Hence by (4.4), one can also parametrise the irreducible modules of $\mathbf{H}_{k,n}$ by the elements of $\sigma(\Phi_{\mathbf{r}}(n))$ and recover the same matrix by labelling the i-th column by $\boldsymbol{\lambda}_i^{\sigma} \in \sigma(\Phi_{\mathbf{r}}(n))$ and the j-th line by $\boldsymbol{\mu}_j^{\sigma} \vdash_l n$.

Moreover, the fact that $\Phi_{\mathbf{r}}(n)$ is the canonical basic set for $(\mathbf{H}_{k,n}, \mathbf{r})$ with respect to $\ll_{\mathbf{m}}$ means that D is upper unitriangular with respect to $\ll_{\mathbf{m}}$. Since we have

$$\boldsymbol{\lambda}_i \ll_{\mathbf{m}} \boldsymbol{\lambda}_j \Leftrightarrow \boldsymbol{\lambda}_i^{\sigma} \ll_{\mathbf{m}^{\sigma}} \boldsymbol{\lambda}_j^{\sigma},$$

the matrix D (with columns indexed by $\sigma(\Phi_{\mathbf{r}}(n))$) is upper unitriangular with respect to $\ll_{\mathbf{m}^{\sigma}}$, i.e. $\sigma(\Phi_{\mathbf{r}}(n))$ is the canonical basic set for $(\mathbf{H}_{k,n}, \mathbf{r}^{\sigma})$.

□

We are now ready to prove the following general result.

Proposition 4.3.5. *Let* $\mathbf{m} = (m_1, ..., m_l) \in \mathbb{Q}^l \backslash \mathscr{P}$. *Then* $(\mathbf{H}_{k,n}, \mathbf{s})$ *admits a canonical basic set with respect to* $\ll_{\mathbf{m}}$, *namely* $\sigma(\Phi_{\mathbf{r}^{\sigma^{-1}}}(n))$, *where* $\mathbf{r}$ *is the* $\mathbf{m}$*-adapted multicharge and* $\sigma \in \mathfrak{S}_l$ *is explicitly determined. We then say that* σ *is the* $\mathbf{m}$*-adapted permutation.*

Proof. As in the level 2 case, consider the $\mathbf{m}$-adapted multicharge $\mathbf{r}$, and set $v_i = r_i - m_i$, for all $i \in [\![1, l]\!]$. Since $\mathbf{m} \notin \mathscr{P}$, at most one coordinate v_i can verify $|v_i| = \frac{e}{2}$. Moreover one can never have $|v_i - v_j| = 0$ for $i \neq j$. Hence, we have $i \neq j \Rightarrow 0 < |v_i - v_j| < e$ for all i, which implies that

$$\text{there exists (a unique) } \tau \in \mathfrak{S}_l \text{ such that } i < j \Rightarrow 0 < v_{\tau(j)} - v_{\tau(i)} < e. \tag{4.5}$$

Since $\mathbf{m} = \mathbf{r} - \mathbf{v}$, we have $\mathbf{m}^\tau = \mathbf{r}^\tau - \mathbf{v}^\tau$. Because of (4.5), we see that $\mathbf{m}^\tau \in \mathcal{D}_{\mathbf{r}^\tau}$, hence by Theorem 4.2.3, $\Phi_{\mathbf{r}^\tau}(n)$ is the canonical basic set for $(\mathbf{H}_{k,n}, \mathbf{r}^\tau)$. Since $\mathbf{r} \in \mathcal{C}_e$, $\mathbf{r}^\tau \in \mathcal{C}_e(\mathbf{s}^\tau)$ and therefore (using Proposition 4.1.1 again), $\Phi_{\mathbf{r}^\tau}(n)$ is the canonical basic set for $(\mathbf{H}_{k,n}, \mathbf{s}^\tau)$ with respect to $\ll_{\mathbf{m}^\tau}$. Thus by Proposition 4.3.4, $\tau^{-1}(\Phi_{\mathbf{r}^\tau}(n))$ is the canonical basic set for $(\mathbf{H}_{k,n}, \mathbf{s})$ with respect to $\ll_{\mathbf{m}}$. Setting $\sigma = \tau^{-1}$, we get the result. □

In the particular level 2 case, we thus have two different approaches which yield canonical basic sets. Let $l = 2$. Let $\mathbf{m} \in \mathbb{Q}^2$, and take $\mathbf{r}$ the $\mathbf{m}$-adapted multicharge. Denote $\sigma = (12)$ (in particular, $\sigma = \sigma^{-1}$). Suppose that $\mathbf{m} \notin \mathcal{D}_{\mathbf{r}}$. On the one hand, by Proposition 4.3.5, $\sigma(\Phi_{\mathbf{r}^\sigma}(n))$ is the canonical basic set for $(\mathbf{H}_{k,n}, \mathbf{s})$. On the other hand, we also have $\mathbf{m} \in \mathcal{D}_{\tilde{\mathbf{r}}}$, so that $\Phi_{\tilde{\mathbf{r}}}(n)$ is the canonical basic set for $(\mathbf{H}_{k,n}, \mathbf{s})$ (this is precisely Proposition 4.3.1).

Hence one must have $\Phi_{\tilde{\mathbf{r}}}(n) = \sigma(\Phi_{\mathbf{r}^\sigma}(n))$. In other terms,

$$\Phi_{(r_1, r_2+e)}(n) = \{(\lambda^2, \lambda^1)\,;\, (\lambda^1, \lambda^2) \in \Phi_{(r_2, r_1)}(n)\}.$$

We recover a result by Jacon, namely [77, Proposition 3.1]. See Example B.0.11 for an illustration of this phenomenon. However, in level $l > 2$, the application $\boldsymbol{\lambda} \longmapsto \boldsymbol{\lambda}^\sigma$ is not necessarily a crystal isomorphism. Consequently, the canonical basic set $\sigma(\Phi_{\mathbf{r}^{\sigma^{-1}}}(n))$ is not a priori a set of Uglov l-partitions. However, we know exactly which of these applications are indeed isomorphisms between sets of some Uglov multipartitions. Indeed, the crystal isomorphisms between the different sets of Uglov multipartitions (associated to $\mathbf{r} \in \mathcal{C}$) have been described by Jacon and Lecouvey in [78]. In particular, [78, Proposition 5.2.1] claims that

$$\Phi_{(r_1, \dots, r_{l-1}, r_l+e)}(n) = \{(\lambda^2, \dots, \lambda^l, \lambda^1)\,;\, (\lambda^1, \dots, \lambda^l) \in \Phi_{(r_l, r_1, \dots, r_{l-1})}(n)\}.$$

This proves that the application

$$\begin{array}{ccc} \Phi_{\mathbf{r}^{\sigma_0^{-1}}}(n) & \longrightarrow & \Phi_{\tilde{\mathbf{r}}}(n) \\ \boldsymbol{\lambda} & \longmapsto & \boldsymbol{\lambda}^{\sigma_0} \end{array} \qquad \text{with } \sigma_0 = (1\ 2\ \dots\ l),$$

where $\tilde{\mathbf{r}} = (r_1, \dots, r_{l-1}, r_l+e)$, is a crystal isomorphism. Applying several times σ_0 (which is of order l), we obtain $l-1$ different crystal isomorphisms $\boldsymbol{\lambda} \longmapsto \boldsymbol{\lambda}^{{\sigma_0}^k}$, $k \in [\![0, l-1]\!]$.

Remark 4.3.6. Note that this application is exactly the inverse of the cyclage introduced in Definition 1.4.3.

Example 4.3.7. Take $l = 3$. Then $\sigma_0 = (1\ 2\ 3)$ and ${\sigma_0}^2 = (1\ 3\ 2)$. Then the following applications are crystal isomorphisms:

$$\begin{array}{ccc} \Phi_{(r_3,r_1,r_2)}(n) & \longrightarrow & \Phi_{(r_1,r_2,r_3+e)}(n) \\ (\lambda^1,\lambda^2,\lambda^3) & \longmapsto & (\lambda^2,\lambda^3,\lambda^1) \end{array},$$

$$\begin{array}{ccc} \Phi_{(r_2,r_3,r_1)}(n) & \longrightarrow & \Phi_{(r_3,r_1,r_2+e)}(n) \\ (\lambda^1,\lambda^2,\lambda^3) & \longmapsto & (\lambda^2,\lambda^3,\lambda^1) \end{array}.$$

We refer to Example B.0.12 for an example of computation of the associated Uglov 3-partitions.

As in the level 2 case, it is possible to recover these results by looking at the domains $\mathcal{D}_{\mathbf{r}}$, $\mathbf{r} \in \mathcal{C}_e$. Indeed, even though these domains do not tile $\mathbb{Q}^l$ (as already mentioned in Remark 4.3.2) some weight sequences $\mathbf{m}$ whose adapted permutation σ verify $\sigma \neq \mathrm{Id}$ can also lie in a domain $\mathcal{D}_{\hat{\mathbf{r}}}$, for some $\hat{\mathbf{r}} \in \mathcal{C}_e$. In that case we have two different constructions of the canonical basic set for $(\mathbf{H}_{k,n}, \mathbf{s})$, which must therefore coincide. That is, for some values of $\sigma \in \mathfrak{S}_l$, the set $\sigma(\Phi_{\mathbf{r}^{\sigma^{-1}}}(n))$ is necessarily a set of Uglov multipartitions $\Phi_{\hat{\mathbf{r}}}(n)$, for some $\hat{\mathbf{r}} \in \mathcal{C}_e$.

Of course, it gets difficult to visualise the domains $\mathcal{D}_{\mathbf{r}}$ when $l \geq 3$. Moreover, this argument does not hold whenever $\boldsymbol{\lambda} \longmapsto \boldsymbol{\lambda}^\sigma$ is not a crystal isomorphism between Uglov multipartitions. In fact, these σ-twisted Uglov multipartitions yield in general new canonical basic sets for $(\mathbf{H}_{k,n}, \mathbf{s})$, since $\sigma(\Phi_{\mathbf{r}^{\sigma^{-1}}}(n))$ is not a set of Uglov l-partitions in general.

4.4 Canonical basic sets for asymptotic weight sequences

Let $\mathbf{r} \in \mathcal{C}_e$. We will show that when the difference between the values of $\mathbf{r}$ is large, the set of Uglov multipartitions stabilises (that is, no longer depends on the parameter $\mathbf{r}$), and coincides with the set of Kleshchev multipartitions. This is what we call the *asymptotic* case, and such an l-tuple $\mathbf{r}$ will be called asymptotic, see Definition 4.4.9.

4.4.1 Kleshchev multipartitions and asymptotic setting

Let us recall in detail the relation between Uglov l-partitions and Kleshchev l-partitions. The Kleshchev l-partitions are defined in the same manner as the Uglov l-partitions, except the order on i-nodes used to define an action of $\mathcal{U}'_q(\widehat{\mathfrak{sl}_e})$ on the Fock space is different. Indeed, let $\gamma = (a, b, c)$ and $\gamma' = (a', b', c')$ be two removable or addable i-nodes of the same l-partition of n. We define

$$\gamma \prec_{\mathcal{K}} \gamma' \Leftrightarrow \begin{cases} c' < c & \text{or} \\ c' = c & \text{and} \quad a' < a. \end{cases}$$

Note that this order only depends on the class $\mathcal{C}_e$, not on some particular $\mathbf{r} \in \mathcal{C}_e$ anymore.

This permits us to give $\mathcal{F}_{\mathbf{r}}$ the structure of integrable $\mathcal{U}'_q(\widehat{\mathfrak{sl}_e})$-module via the same formulas used with $\prec_{\mathbf{r}}$ in Theorem 3.2.2. We can then construct the crystal graph of the highest weight submodule spanned by $|\boldsymbol{\emptyset}, \mathbf{r}\rangle$, in the same way as the Uglov multipartitions (cf. Theorem 3.2.10 and Corollary 3.2.12). Its vertices are labeled by what we call the *Kleshchev l-partitions*. We denote by $\mathcal{K}_{\mathcal{C}_e}(n)$ the set of Kleshchev l-partitions of rank n.

Note that with this realisation as an $\mathcal{U}'_q(\widehat{\mathfrak{sl}_e})$-module, $\mathcal{F}_{\mathbf{r}}$ is actually a tensor product of Fock spaces of level 1, see [126].

The following proposition connects both orders for certain values of $\mathbf{r}$.

Proposition 4.4.1. *Let $\mathbf{r} \in \mathcal{C}_e$ such that $i < j \Rightarrow r_i - r_j \geq n - e + 1$* [1]*. Then for all $m \leq n$, $\Phi_{\mathbf{r}}(m) = \mathcal{K}_{\mathcal{C}_e}(m)$. In particular , $\Phi_{\mathbf{r}}(n) = \mathcal{K}_{\mathcal{C}_e}(n)$.*

Proof. It is sufficient to show that in this case, both orders on i-nodes are equivalent, i.e. $\gamma \prec_{\mathbf{r}} \gamma' \Leftrightarrow \gamma \prec_{\mathcal{K}} \gamma'$, where $\gamma = (a, b, c)$ and $\gamma' = (a', b', c')$ are two removable or addable i-nodes of $\boldsymbol{\lambda} \vdash_l m$.

Note that $-n \leq b' - a' - (b - a) \leq n$. Indeed, the difference between $b' - a'$ and $b - a$ is minimal if and only if ($\lambda^{c'} = \emptyset$ and $\lambda^c = (n)$) or ($\lambda^{c'} = (1^n)$ and $\lambda^c = \emptyset$); and is maximal if and only if ($\lambda^{c'} = (n)$ and $\lambda^c = \emptyset$) or ($\lambda^{c'} = \emptyset$ and $\lambda^c = (1^n)$).

First assume that $\gamma \prec_{\mathcal{K}} \gamma'$. Then:

- If $c' < c$, then $r_{c'} - r_c \geq n - e + 1$, hence $b' - a' + r_{c'} - (b - a + r_c) \geq -n + n - e + 1 = -e + 1$. Since γ and γ' have the same residue, this implies that $b' - a' + r_{c'}$ and $b - a + r_c$ are congruent modulo e, thus $b' - a' + r_{c'} - (b - a + r_c) \geq 0$, and therefore $\gamma \prec_{\mathbf{r}} \gamma'$.
- If $c' = c$ and $a' < a$, then $b < b'$ since $\lambda^{c'} = \lambda^c$ is a partition and γ and γ' are on the border of λ^c. Hence $b - a < b' - a'$, and $b - a + r_c < b' - a' + r_{c'}$, hence $\gamma \prec_{\mathbf{r}} \gamma'$.

Conversely, assume that $\gamma \prec_{\mathbf{r}} \gamma'$. Then:

- If $b - a + r_c < b' - a' + r_{c'}$ then suppose $c' > c$. Then $r_c - r_{c'} \geq n - e + 1$. Since γ and γ' have the same residue, we have $b' - a' + r_{c'} - (b - a + r_c) \geq e$, and thus $b' - a' - (b - a) \geq e + n - e + 1 = n + 1$, whence a contradiction. Hence $c' \leq c$. If $c' < c$ then $\gamma \prec_{\mathcal{K}} \gamma'$, and if $c' = c$ then $b' - a' > b - a$ thus $a' < a$ for the same reason as before, and $\gamma \prec_{\mathcal{K}} \gamma'$.
- If $b - a + r_c = b' - a' + r_{c'}$ and $c' < c$ then it is straightforward that $\gamma \prec_{\mathcal{K}} \gamma'$.

The only difference (a priori) in the construction of the Uglov l-partitions on the one hand, and the Kleshchev l-partitions on the other hand is the definition of the order on i-nodes. Since we just proved that both orders coincide in this case, both sets are the same.

$\square$

[1] Of course, this is equivalent to $r_i - r_{i+1} \geq n - e + 1$ for all $i \in [\![1, l-1]\!]$.

From this Proposition, we directly deduce:

Corollary 4.4.2. *Suppose $\mathbf{r} \in \mathcal{C}_e$. When the difference $r_i - r_j$, for all $i < j$, is sufficiently large, the set of Uglov multipartitions $\Phi_{\mathbf{r}}(n)$ stabilises, and is equal to $\mathcal{K}_{\mathcal{C}_e}(n)$.*

Remark 4.4.3. Note that the bound $n - e + 1$ is not necessarily sharp (even though it is an optimal condition for both orders on i-nodes to coincide), it is a priori possible for Uglov multipartitions to stabilise at a weaker condition on $\mathbf{r}$.

Actually, the set of Uglov multipartitions stabilises in other directions, that is, under other conditions of $\mathbf{r}$. More precisely, we will show that they stabilise whenever the difference between any two arbitrary coordinates of $\mathbf{r}$ (without the condition $i < j$) is "large enough".

In order to describe this phenomenon, we introduce the set of *twisted Kleshchev multipartitions*. Let $\pi \in \mathfrak{S}_l$. We define the *π-twisted Kleshchev order* on i-nodes as follows: Let $\gamma = (a, b, c)$ and $\gamma' = (a', b', c')$ be two removable or addable i-nodes of the same l-partition of n. We set

$$\gamma \prec_{\mathcal{K}}^{\pi} \gamma' \Leftrightarrow \begin{cases} \pi(c') < \pi(c) & \text{or} \\ \pi(c') = \pi(c) & \text{and} \quad a' < a \end{cases}$$

This just means that the lexicographic convention on the coordinates of the l-partition is twisted by π. The *π-twisted Kleshchev l-partitions* are then defined as in the non-twisted case (and as the in the "Uglov" case): they label the vertices of the crystal graph of the same highest weight module defined via the action of $\mathcal{U}_q'(\widehat{\mathfrak{sl}_e})$ derived from this order $\prec_{\mathcal{K}}^{\pi}$. We denote them by $\mathcal{K}_{\mathcal{C}_e}^{\pi}(n)$.

Remark 4.4.4. Note that it is equivalent to either build the set of π-twisted Kleshchev multipartitions associated to $\mathcal{C}_e$, or to twist via π the set of Kleshchev multipartitions associated to $\mathcal{C}_e(\mathbf{s}^{\pi^{-1}})$, i.e.

$$\mathcal{K}_{\mathcal{C}_e}^{\pi}(n) = \pi(\mathcal{K}_{\mathcal{C}_e^{\pi^{-1}}}(n)),$$

where $\mathcal{C}_e^{\pi^{-1}} := \mathcal{C}_e(\mathbf{s}^{\pi^{-1}})$.

We have the following "asymptotic" property:

Proposition 4.4.5. *Let $\mathbf{s} \in \mathcal{C}_e$ such that there exists $\pi \in \mathfrak{S}_l$ verifying $\pi(i) < \pi(j) \Rightarrow s_i - s_j \geq n + 1$. Then $\Phi_{\mathbf{r}}(m) = \mathcal{K}_{\mathcal{C}_e}^{\pi}(m)$ for all $m \leq n$. In particular, $\Phi_{\mathbf{r}}(n) = \mathcal{K}_{\mathcal{C}_e}^{\pi}(n)$.*

Proof. It is very similar to the one of Proposition 4.4.1. Indeed, we show that for $\gamma = (a, b, c)$ and $\gamma' = (a', b', c')$ two removable or addable i-nodes of $\boldsymbol{\lambda} \vdash_l m$, $\gamma \prec_{\mathbf{r}} \gamma' \Leftrightarrow \gamma \prec_{\mathcal{K}}^{\pi} \gamma'$.

Assume that $\gamma \prec_{\mathcal{K}}^{\pi} \gamma'$. Then:

- If $\pi(c') < \pi(c)$, then $r_{c'} - r_c \geq n+1$, hence $b' - a' + r_{c'} - (b - a + r_c) \geq -n + n + 1 = 1 > 0$. Hence $b' - a' + r_{c'} > (b - a + r_c)$ and $\gamma \prec_{\mathbf{r}} \gamma'$.

- If $\pi(c') = \pi(c)$ and $a' < a$. Then $c' = c$ since π is a permutation. Thus $b < b'$ since $\lambda^{c'} = \lambda^c$ is a partition and γ and γ' are on the border of λ^c. Hence $b - a < b' - a'$, and $b - a + r_c < b' - a' + r_{c'}$, hence $\gamma \prec_{\mathbf{r}} \gamma'$.

Conversely, assume that $\gamma \prec_{\mathbf{r}} \gamma'$. Then:

- If $b - a + r_c < b' - a' + r_{c'}$ then suppose $\pi(c') > \pi(c)$. Then $s_c - s_{c'} \geq n + 1$, and thus $b' - a' - (b - a) > n + 1$, whence a contradiction. Hence $\pi(c') \leq \pi(c)$. If $\pi(c') < \pi(c)$ then $\gamma \prec_{\mathcal{K}}^{\pi} \gamma'$, and if $\pi(c') = \pi(c)$ then $c' = c$ and $b' - a' > b - a$ thus $a' < a$, and $\gamma \prec_{\mathcal{K}}^{\pi} \gamma'$.
- If $b - a + r_c = b' - a' + r_{c'}$ and $\pi(c') < \pi(c)$ then $\gamma \prec_{\mathcal{K}}^{\pi} \gamma'$.

Again, the only difference in the construction of the Uglov l-partitions on the one hand, and the π-twisted Kleshchev l-partitions on the other hand is the definition of the order on i-nodes. Since we have proved that both orders coincide in this case, these sets are the same.

□

Hence, we directly deduce the following stabilisation property, whenever the difference between two arbitrary coordinates of $\mathbf{r}$ is large:

Corollary 4.4.6. *Let $\mathbf{r} \in \mathcal{C}_e$ and let $\pi \in \mathfrak{S}_l$. When the difference $r_i - r_j$, for all $\pi(i) < \pi(j)$, is sufficiently large, then the set of Uglov l-partitions $\Phi_{\mathbf{r}}(n)$ stabilises, and is equal to $\mathcal{K}_{\mathcal{C}_e}^{\pi}(n)$.*

Note that such a permutation π verifies in particular $\pi(i) < \pi(j) \Rightarrow r_i > r_j$. We thus call π the *reordering permutation of* $\mathbf{r}$.

Remark 4.4.7. As in Remark 4.4.3, note that the bound is not necessarily sharp, and that Uglov multipartitions are likely to stabilise under weaker conditions. In fact, when $\pi = \mathrm{Id}$, Proposition 4.4.5 gives a bound (namely $n+1$) on each $r_i - r_j$ beyond which $\Phi_{\mathbf{r}}(n) = \mathcal{K}_{\mathcal{C}_e}(n)$, but which is less precise than the one given in Proposition 4.4.1 (namely $n - e + 1$). However, when $\pi \neq \mathrm{Id}$, the bound $n + 1$ is optimal for the orders $\prec_{\mathbf{r}}$ and $\prec_{\mathcal{K}}^{\pi}$ to coincide.

Remark 4.4.8. Let $\mathbf{r}$ and π be as in Corollary 4.4.6, i.e. $\Phi_{\mathbf{r}}(n) = \mathcal{K}_{\mathcal{C}_e}^{\pi}(n)$.

It is important to notice that for all $\sigma \in \mathfrak{S}_l$,

$$\Phi_{\mathbf{r}^\sigma}(n) = \sigma(\mathcal{K}_{\mathcal{C}_e}^{\pi}(n)). \tag{4.6}$$

Indeed, this directly follows from the definition of the Kleshchev order on i-nodes. Since in this case $\Phi_{\mathbf{r}}(n)$ is a set of (π-twisted) Kleshchev multipartitions, it is equivalent to either

- twist the multicharge via $\mathbf{r} \mapsto \mathbf{r}^\sigma$ and build the corresponding Uglov crystal, or

- twist via $\boldsymbol{\lambda} \mapsto \boldsymbol{\lambda}^\sigma$ these π-twisted Kleshchev l-partitions.

In other terms, replacing σ by σ^{-1}, (4.6) is equivalent to:

$$\sigma(\Phi_{\mathbf{r}^{\sigma^{-1}}}(n)) = \mathcal{K}^{\pi}_{\mathcal{C}_e}(n).$$

In particular, this shows that the canonical basic set $\sigma(\Phi_{\mathbf{r}^{\sigma^{-1}}}(n))$ of Proposition 4.3.5 is always equal to $\mathcal{K}^{\pi}_{\mathcal{C}_e}(n)$, for any value of $\sigma \in \mathfrak{S}_l$ when $\Phi_{\mathbf{r}}(n) = \mathcal{K}^{\pi}_{\mathcal{C}_e}(n)$.

We can now define *asymptotic* multicharges and weight sequences.

Definition 4.4.9.

1. Let $\mathbf{r} \in \mathcal{C}_e$. We say that $\mathbf{r}$ is asymptotic if $\Phi_{\mathbf{r}}(n) = \mathcal{K}^{\pi}_{\mathcal{C}_e}(n)$ for some $\pi \in \mathfrak{S}_l$ (in which case π is the reordering permutation of $\mathbf{r}$).

2. Let $\mathbf{m} \in \mathbb{Q}^l$. We say that $\mathbf{m}$ is asymptotic if the $\mathbf{m}$-adapted multicharge (see Proposition 4.3.5) is asymptotic.

Remark 4.4.10. According to Remark 4.4.8, $\mathbf{r}$ is asymptotic if and only if for all $\sigma \in \mathfrak{S}_l$, $\sigma(\Phi_{\mathbf{r}^{\sigma^{-1}}}(n)) = \mathcal{K}^{\pi}_{\mathcal{C}_e}(n)$.

Let us now focus on the question of the existence of canonical basic sets, given an asymptotic weight sequence $\mathbf{m}$. In the case where $\mathbf{m} \notin \mathscr{P}$, $\mathbf{m}$ is regular, and we have already shown in Proposition 4.3.5 that $(\mathbf{H}_{k,n}, \mathbf{s})$ admits a canonical basic set with respect to $\ll_{\mathbf{m}}$, namely the set $\sigma(\Phi_{\mathbf{r}^{\sigma^{-1}}}(n))$ where $\mathbf{r}$ is the $\mathbf{m}$-adapted multicharge and σ the $\mathbf{m}$-adapted permutation. In virtue of Remark 4.4.8, these sets of l-partitions are all equal to $\mathcal{K}^{\pi}_{\mathcal{C}_e}(n)$. We will show that in the remaining asymptotic cases, the order $\ll_{\mathbf{m}}$ yields a canonical basic set for $(\mathbf{H}_{k,n}, \mathbf{s})$ which is also a set of twisted Kleshchev multipartitions.

4.4.2 Kleshchev multipartitions as canonical basic sets

Fix $\mathbf{m} \in \mathscr{P}$ such that $\mathbf{m}$ is asymptotic. Because of Lemma 4.2.5, this means that $\mathbf{m} \in \bigcup_{(i,j)\in J} \mathcal{P}_{i,j}(\mathbf{r})$, where $\mathbf{r}$ is the $\mathbf{m}$-adapted multicharge and is asymptotic, and where $J \subset [\![1, l]\!]^2$.

In order to understand the phenomenon that happens, it is interesting to keep in mind the results of Uglov. Recall that he has endowed the Fock space with a so-called canonical basis (Theorem 3.2.14). By keeping the elements indexed by the Uglov l-partitions, one recovers the canonical basis of $V(\mathbf{s})$ (Theorem 3.2.16), which enables the computation of the decomposition matrix D of $\mathbf{H}_{k,n}$ by Ariki's Theorem, by specialising q at 1 (Theorem 3.3.1). He showed ([126, Proposition 4.11]) that the matrix of the canonical basis of $\mathcal{F}_{\mathbf{s}}$ is always unitriangular with respect to the combinatorial order $\leq_{\mathcal{U}}$ (see Section 1.2.3). By restriction, this implies that the matrix of the canonical basis of $V(\mathbf{s})$, and, in turn, the decomposition matrix D, are both unitriangular with respect to $\leq_{\mathcal{U}}$. This implies that $(\mathbf{H}_{k,n}, \mathbf{s})$ has a canonical basic set with respect to $\leq_{\mathcal{U}}$, namely the set of Uglov

l-partitions. Now, when $\mathbf{r}$ is "asymptotic enough", one has the compatibility propery stated in Proposition 1.2.11:

$$\boldsymbol{\lambda} \leq_{\mathcal{U}} \boldsymbol{\mu} \Rightarrow \boldsymbol{\mu} \ll_{\mathbf{s}} \boldsymbol{\lambda}.$$

Because $\mathbf{r}$ is the $\mathbf{m}$-adapted multicharge, $\mathbf{r}$ and $\mathbf{m}$ are "close", and one can replace the order $\ll_{\mathbf{r}}$ by $\ll_{\mathbf{m}}$.

$$\boldsymbol{\lambda} \leq_{\mathcal{U}} \boldsymbol{\mu} \Rightarrow \boldsymbol{\mu} \ll_{\mathbf{m}} \boldsymbol{\lambda}. \tag{4.7}$$

Thus, if $\mathbf{m} \in \mathcal{P}_{i,j}(\mathbf{r})$ for such an $\mathbf{r}$, we are ensured that the set of Uglov multipartitions (which coincide with some π-twisted Kleshchev multipartitions) is the canonical basic set for $(\mathbf{H}_{k,n}, \mathbf{s})$ with respect to $\ll_{\mathbf{m}}$.

Unfortunately, this particular setting does not cover all the asymptotic cases. Indeed, the definition of an asymptotic weight sequence given in 4.4.9 is not sufficient to deduce the compatibility property (4.7) above. However, using only combinatorial arguments, we can show the more general following result.

Proposition 4.4.11. *Let $\mathbf{m} \in \mathscr{P}$ be an asymptotic weight sequence, let $\mathbf{r}$ be the $\mathbf{m}$-adapted multicharge. Then $(\mathbf{H}_{k,n}, \mathbf{s})$ admits a canonical basic set with respect to $\ll_{\mathbf{m}}$, namely the set $\mathcal{K}^{\pi}_{\mathcal{C}_e}(n)$, where π is the reordering permutation of $\mathbf{r}$.*

In order to prove this, we need the following technical lemma. We introduce the following notation. Given a weight sequence $\mathbf{m}$, $\varepsilon > 0$ and $I \subset [\![1,l]\!]$, we define a new weight sequence by $\mathbf{m}^{[\varepsilon,I]} := (m_i^{[\varepsilon,I]})_{i=1\dots l}$ where

$$m_i^{[\varepsilon,I]} = \begin{cases} m_i \text{ if } i \notin I \\ m_i + i\varepsilon \text{ if } i \in I \end{cases}$$

Example 4.4.12. Take $l = 3$ and $I = \{1,3\}$. Then $\mathbf{m}^{[\varepsilon,I]} = (m_1 + \varepsilon, m_2, m_3 + 3\varepsilon)$.

Lemma 4.4.13. *Let $\mathbf{m}$ be an arbitrary weight sequence. Let $\boldsymbol{\lambda}, \boldsymbol{\mu} \vdash_l n$, $\boldsymbol{\lambda} \neq \boldsymbol{\mu}$. Then there exists $\alpha_{\boldsymbol{\lambda},\boldsymbol{\mu}} > 0$ such that for all $\varepsilon \in]0, \alpha_{\boldsymbol{\lambda},\boldsymbol{\mu}}[$,*

1.

$$\boldsymbol{\lambda} \ll_{\mathbf{m}} \boldsymbol{\mu} \quad \Rightarrow \quad \left[\begin{array}{c} \forall I \subset [\![1,l]\!],\ \textit{either} \qquad \boldsymbol{\lambda} \ll_{\mathbf{m}^{[\varepsilon,I]}} \boldsymbol{\mu} \qquad \textit{or} \\ \boldsymbol{\lambda} \textit{ and } \boldsymbol{\mu} \textit{ are not comparable with respect to } \ll_{\mathbf{m}^{[\varepsilon,I]}}, \end{array} \right]$$

and

2.

$$\left[\begin{array}{c} \boldsymbol{\lambda} \textit{ and } \boldsymbol{\mu} \textit{ are not comparable} \\ \textit{with respect to } \ll_{\mathbf{m}} \end{array} \right] \Rightarrow \left[\begin{array}{c} \boldsymbol{\lambda} \textit{ and } \boldsymbol{\mu} \textit{ are not comparable with} \\ \textit{respect to } \ll_{\mathbf{m}^{[\varepsilon,I]}},\ \forall I \subset [\![1,l]\!]. \end{array} \right],$$

This means that for a small perturbation of $\mathbf{m}$, the order $\ll_{\mathbf{m}}$ never reverses: at worst, $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ become uncomparable. Moreover, one can never gain comparability between multipartitions uncomparable with respect to $\ll_{\mathbf{m}}$ when slightly perturbing $\mathbf{m}$.

Proof. First, note that it is sufficient to prove these properties for the perturbations $\mathbf{m}^{[\varepsilon,k]}$ of $\mathbf{m}$ defined by $\mathbf{m}^{[\varepsilon,k]}=(m_1,\dots,m_{k-1},m_k+\varepsilon,m_{k+1},\dots,m_l)$, for all $k\in[\![1,l]\!]$. Indeed, the result then follows by induction, since $\mathbf{m}^{[\varepsilon,I]}$ is constructed by iterating this procedure.

Recall that, by definition (Section 1.2), $\boldsymbol{\lambda}\ll_{\mathbf{m}}\boldsymbol{\mu}$ and $\boldsymbol{\lambda}\neq\boldsymbol{\mu}$ means that $\mathfrak{b}_{\mathbf{m}}(\boldsymbol{\lambda})\rhd\mathfrak{b}_{\mathbf{m}}(\boldsymbol{\mu})$, where $\mathfrak{b}_{\mathbf{m}}(\lambda)=(\mathfrak{b}^1_{\mathbf{m}}(\boldsymbol{\lambda}),\dots,\mathfrak{b}^h_{\mathbf{m}}(\boldsymbol{\lambda}))$ is the decreasing sequence consisting of the elements of $\mathfrak{B}_{\mathbf{m}}(\boldsymbol{\lambda})$.

Let us first prove Assertion 1.

Let $\boldsymbol{\lambda}\ll_{\mathbf{m}}\boldsymbol{\mu}$. For $k\in[\![1,l]\!]$, consider the order $\ll_{\mathbf{m}^{[\varepsilon,k]}}$. It is obtained from $\ll_{\mathbf{m}}$ simply by translating the k-th row of the symbols by ε, and taking the dominance order on the decreasing sequences of these new elements. Informally, when we choose ε to be "small", one cannot have $\boldsymbol{\mu}\ll_{\mathbf{m}^{[\varepsilon,k]}}\boldsymbol{\lambda}$. Indeed, the fact that $\boldsymbol{\lambda}\ll_{\mathbf{m}}\boldsymbol{\mu}$ creates a gap at some point between $\sum_i\mathfrak{b}^i_{\mathbf{m}}(\boldsymbol{\mu})$ and $\sum_i\mathfrak{b}^i_{\mathbf{m}}(\boldsymbol{\lambda})$, which cannot be recovered if ε is small enough.

Let us prove this properly. We also use a running example to illustrate the different points of the coming proof. For simplicity, since $\mathbf{m}\in\mathcal{P}_{i,j}(\mathbf{r})$ for some i,j and some $\mathbf{r}\in\mathbb{Z}^l$, we can assume without loss of generality that m_i and m_j are integers.

Example: Take $e=2$, $l=3$, $n=38$, $\mathbf{s}=(1,0,0)$, $\mathbf{m}=(3{}^1\!/\!_3,7,5)\in\mathcal{P}_{2,3}(1,2,0)$. Let $\boldsymbol{\lambda}=(4.1,4^2.3.2^3.1,4^3.2.1)\vdash_l n$ and $\boldsymbol{\mu}=(4.2,4.3.2^5,5.4^2.1^2)\vdash_l n$. The shifted $\mathbf{m}$-symbols of $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ of size 1 are the following:

$$\mathfrak{B}_{\mathbf{m}}(\boldsymbol{\lambda})=\begin{pmatrix}0&2&4&7&8&9&&\\0&2&4&5&6&8&10&11\\0{}^1\!/\!_3&1{}^1\!/\!_3&3{}^1\!/\!_3&7{}^1\!/\!_3&&&&\end{pmatrix}$$

and

$$\mathfrak{B}_{\mathbf{m}}(\boldsymbol{\mu})=\begin{pmatrix}0&2&3&7&8&10&&\\0&3&4&5&6&7&9&11\\0{}^1\!/\!_3&1{}^1\!/\!_3&4{}^1\!/\!_3&7{}^1\!/\!_3&&&&\end{pmatrix}.$$

The corresponding sequences $\mathfrak{b}_{\mathbf{m}}$ are

$$\mathfrak{b}_{\mathbf{m}}(\boldsymbol{\lambda})=(11,10,9,8,8,7{}^1\!/\!_3,7,6,5,4,4,3{}^1\!/\!_3,2,2,1{}^1\!/\!_3,0{}^1\!/\!_3,0,0)$$

and

$$\mathfrak{b}_{\mathbf{m}}(\boldsymbol{\mu})=(11,10,9,8,7{}^1\!/\!_3,7,7,6,5,4{}^1\!/\!_3,4,3,3,2,1{}^1\!/\!_3,0{}^1\!/\!_3,0,0).$$

Since $\boldsymbol{\lambda}\ll_{\mathbf{m}}\boldsymbol{\mu}$ and $\boldsymbol{\lambda}\neq\boldsymbol{\mu}$, there exists a smallest integer p such that $\mathfrak{b}^p_{\mathbf{m}}(\boldsymbol{\lambda})>\mathfrak{b}^p_{\mathbf{m}}(\boldsymbol{\mu})$. Denote $\delta=\mathfrak{b}^p_{\mathbf{m}}(\boldsymbol{\lambda})-\mathfrak{b}^p_{\mathbf{m}}(\boldsymbol{\mu})$.

In our example, $p=5$ and $\delta=2/3$, since $\mathfrak{b}^i_{\mathbf{m}}(\boldsymbol{\lambda})=\mathfrak{b}^i_{\mathbf{m}}(\boldsymbol{\mu})\,\forall i<5$ and $\mathfrak{b}^5_{\mathbf{m}}(\boldsymbol{\lambda})=8$ and $\mathfrak{b}^5_{\mathbf{m}}(\boldsymbol{\mu})=7{}^1\!/\!_3$.

Now for all i, denote $\{m_i\}=m_i-\lfloor m_i\rfloor$ the fractional part of m_i, whenever $m_i\notin\mathbb{Z}$. Set $\beta_i=\min(\{m_i\},1-\{m_i\})$ (for all $m_i\notin\mathbb{Z}$), and $\beta=\min_i\beta_i$. If $m_i\in\mathbb{Z}$ for all i, then set $\beta=1$. In particular $\beta\leq\delta$. In the example, $\beta=1/3$.

Hence, set $0 < \varepsilon < \beta$. Now, for all $k \in [\![1, l]\!]$, consider the $\mathbf{m}^{[\varepsilon,k]}$-symbols of $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$. In our example, for $k = 3$ for instance, we get

$$\mathfrak{B}_{\mathbf{m}^{[\varepsilon,1]}}(\boldsymbol{\lambda}) = \begin{pmatrix} 0+\varepsilon & 2+\varepsilon & 4+\varepsilon & 7+\varepsilon & 8+\varepsilon & 9+\varepsilon & & \\ 0 & 2 & 4 & 5 & 6 & 8 & 10 & 11 \\ 0^1\!/3 & 1^1\!/3 & 3^1\!/3 & 7^1\!/3 & & & & \end{pmatrix}$$

and

$$\mathfrak{B}_{\mathbf{m}^{[\varepsilon,1]}}(\boldsymbol{\mu}) = \begin{pmatrix} 0+\varepsilon & 2+\varepsilon & 3+\varepsilon & 7+\varepsilon & 8+\varepsilon & 10+\varepsilon & & \\ 0 & 3 & 4 & 5 & 6 & 7 & 9 & 11 \\ 0^1\!/3 & 1^1\!/3 & 4^1\!/3 & 7^1\!/3 & & & & \end{pmatrix}.$$

Since $\varepsilon < \beta$, the "perturbed" elements (of $\mathfrak{B}_{\mathbf{m}^{[\varepsilon,k]}}$) are ordered in the same way as the original ones (those of $\mathfrak{B}_{\mathbf{m}}$). Precisely, for all i, we either have

$$\mathfrak{b}^i_{\mathbf{m}^{[\varepsilon,k]}}(\boldsymbol{\lambda}) = \begin{cases} \mathfrak{b}^i_{\mathbf{m}}(\boldsymbol{\lambda}) & \text{or} \\ \mathfrak{b}^i_{\mathbf{m}}(\boldsymbol{\lambda}) + \varepsilon, & \end{cases} \tag{4.8}$$

and similarly for $\boldsymbol{\mu}$.

Now, let $\alpha_{\boldsymbol{\lambda},\boldsymbol{\mu}} = \min(\beta, \delta/p)$ and take $0 < \varepsilon < \alpha_{\boldsymbol{\lambda},\boldsymbol{\mu}}$. One can then compute $\sum_{i=1}^s \mathfrak{b}^i_{\mathbf{m}^{[\varepsilon,k]}}(\boldsymbol{\lambda})$ and $\sum_{i=1}^s \mathfrak{b}^i_{\mathbf{m}^{[\varepsilon,k]}}(\boldsymbol{\mu})$for all $s < p$. Clearly, it is possible to have $\sum_{i=1}^s \mathfrak{b}^i_{\mathbf{m}^{[\varepsilon,k]}}(\boldsymbol{\lambda}) < \sum_{i=1}^s \mathfrak{b}^i_{\mathbf{m}^{[\varepsilon,k]}}(\boldsymbol{\mu})$. This is the case in the example, for $k = 3$, since if we take $s = 2$, we have $\mathfrak{b}^1_{\mathbf{m}^{[\varepsilon,1]}}(\boldsymbol{\lambda}) + \mathfrak{b}^2_{\mathbf{m}^{[\varepsilon,1]}}(\boldsymbol{\lambda}) = 11 + 10 < 11 + 10 + \varepsilon = \mathfrak{b}^1_{\mathbf{m}^{[\varepsilon,1]}}(\boldsymbol{\mu}) + \mathfrak{b}^2_{\mathbf{m}^{[\varepsilon,1]}}(\boldsymbol{\mu})$. Hence, one can have $\boldsymbol{\lambda} \not\ll_{\mathbf{m}^{[\varepsilon,k]}} \boldsymbol{\mu}$.

However, we necessarily have:

- $\displaystyle\sum_{i=1}^{p-1} \mathfrak{b}^i_{\mathbf{m}^{[\varepsilon,k]}}(\boldsymbol{\mu}) - \sum_{i=1}^{p-1} \mathfrak{b}^i_{\mathbf{m}^{[\varepsilon,k]}}(\boldsymbol{\lambda}) \leq (p-1)\varepsilon$, and
- $\mathfrak{b}^p_{\mathbf{m}^{[\varepsilon,k]}}(\boldsymbol{\lambda}) - \mathfrak{b}^p_{\mathbf{m}^{[\varepsilon,k]}}(\boldsymbol{\mu}) \geq \delta - \varepsilon$ since $\mathfrak{b}^p_{\mathbf{m}}(\boldsymbol{\lambda}) - \mathfrak{b}^p_{\mathbf{m}}(\boldsymbol{\mu}) = \delta$.

Thus,

$$\begin{aligned}\sum_{i=1}^{p} \mathfrak{b}^i_{\mathbf{m}^{[\varepsilon,k]}}(\boldsymbol{\lambda}) - \sum_{i=1}^{p} \mathfrak{b}^i_{\mathbf{m}^{[\varepsilon,k]}}(\boldsymbol{\mu}) &\geq -(p-1)\varepsilon + \delta - \varepsilon \\ &= -p\varepsilon + \delta \\ &> -p\frac{\delta}{p} + \delta \quad \text{since} \quad \varepsilon < \frac{\delta}{p} \\ &= 0.\end{aligned}$$

Hence one can never have $\boldsymbol{\mu} \ll_{\mathbf{m}^{[\varepsilon,k]}} \boldsymbol{\lambda}$, which proves the first point.

The proof of Assertion 2. is completely similar. First, if $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are not comparable with respect to $\ll_{\mathbf{m}}$, then there exist minimal integers p_1 and p_2 such that

$$\sum_{i=1}^{p_1} \mathfrak{b}^i_{\mathbf{m}}(\boldsymbol{\lambda}) > \sum_{i=1}^{p_1} \mathfrak{b}^i_{\mathbf{m}}(\boldsymbol{\mu}) \quad \text{and} \quad \sum_{i=1}^{p_2} \mathfrak{b}^i_{\mathbf{m}}(\boldsymbol{\lambda}) < \sum_{i=1}^{p_2} \mathfrak{b}^i_{\mathbf{m}}(\boldsymbol{\mu}).$$

We can assume without loss of generality that $p_1 < p_2$. We denote $\delta_1 = \mathfrak{b}^{p_1}_{\mathbf{m}}(\boldsymbol{\lambda}) - \mathfrak{b}^{p_1}_{\mathbf{m}}(\boldsymbol{\mu}) > 0$ and $\delta_2 = \mathfrak{b}^{p_2}_{\mathbf{m}}(\boldsymbol{\mu}) - \mathfrak{b}^{p_2}_{\mathbf{m}}(\boldsymbol{\lambda}) > 0$. We take $\alpha_{\boldsymbol{\lambda},\boldsymbol{\mu}} = \min\{\beta, \delta_1/p_1, \delta_2/p_2\}$, where β is as in the proof of Assertion 1. Again, because $\varepsilon < \beta$, we know that

$$\mathfrak{b}^i_{\mathbf{m}^{[\varepsilon,k]}}(\boldsymbol{\lambda}) = \begin{cases} \mathfrak{b}^i_{\mathbf{m}}(\boldsymbol{\lambda}) & \text{or} \\ \mathfrak{b}^i_{\mathbf{m}}(\boldsymbol{\lambda}) + \varepsilon, \end{cases}$$

Now, on the one hand, we have

- $\displaystyle\sum_{i=1}^{p_1-1} \mathfrak{b}^i_{\mathbf{m}^{[\varepsilon,k]}}(\boldsymbol{\mu}) - \sum_{i=1}^{p_1-1} \mathfrak{b}^i_{\mathbf{m}^{[\varepsilon,k]}}(\boldsymbol{\lambda}) \leq (p_1 - 1)\varepsilon$, and
- $\mathfrak{b}^{p_1}_{\mathbf{m}^{[\varepsilon,k]}}(\boldsymbol{\lambda}) - \mathfrak{b}^{p_1}_{\mathbf{m}^{[\varepsilon,k]}}(\boldsymbol{\mu}) \geq \delta_1 - \varepsilon$ since $\mathfrak{b}^{p_1}_{\mathbf{m}}(\boldsymbol{\lambda}) - \mathfrak{b}^{p_1}_{\mathbf{m}}(\boldsymbol{\mu}) = \delta_1$.

This gives

$$\begin{array}{rcl} \displaystyle\sum_{i=1}^{p_1} \mathfrak{b}^i_{\mathbf{m}^{[\varepsilon,k]}}(\boldsymbol{\lambda}) - \sum_{i=1}^{p_1} \mathfrak{b}^i_{\mathbf{m}^{[\varepsilon,k]}}(\boldsymbol{\mu}) & \geq & -(p_1 - 1)\varepsilon + \delta_1 - \varepsilon \\ & = & -p_1\varepsilon + \delta_1 \\ & > & -p_1\frac{\delta_1}{p_1} + \delta_1 \quad \text{since} \quad \varepsilon < \frac{\delta_1}{p_1} \\ & = & 0. \end{array}$$

On the other hand, we have

- $\displaystyle\sum_{i=1}^{p_2-1} \mathfrak{b}^i_{\mathbf{m}^{[\varepsilon,k]}}(\boldsymbol{\lambda}) - \sum_{i=1}^{p_2-1} \mathfrak{b}^i_{\mathbf{m}^{[\varepsilon,k]}}(\boldsymbol{\mu}) \leq (p_2 - 1)\varepsilon$, and
- $\mathfrak{b}^{p_2}_{\mathbf{m}^{[\varepsilon,k]}}(\boldsymbol{\lambda}) - \mathfrak{b}^{p_2}_{\mathbf{m}^{[\varepsilon,k]}}(\boldsymbol{\mu}) \leq -\delta_2 + \varepsilon$ since $\mathfrak{b}^{p_2}_{\mathbf{m}}(\boldsymbol{\mu}) - \mathfrak{b}^{p_2}_{\mathbf{m}}(\boldsymbol{\lambda}) = \delta_2$.

This gives

$$\begin{array}{rcl} \displaystyle\sum_{i=1}^{p_2} \mathfrak{b}^i_{\mathbf{m}^{[\varepsilon,k]}}(\boldsymbol{\lambda}) - \sum_{i=1}^{p_2} \mathfrak{b}^i_{\mathbf{m}^{[\varepsilon,k]}}(\boldsymbol{\mu}) & \leq & (p_2 - 1)\varepsilon + (-\delta_2 + \varepsilon) \\ & = & p_2\varepsilon - \delta_2 \\ & < & p_2\frac{\delta_2}{p_2} - \delta_2 \quad \text{since} \quad \varepsilon < \frac{\delta_2}{p_2} \\ & = & 0. \end{array}$$

This implies in particular that $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are not comparable with respect to the perturbed order $\ll_{\mathbf{m}^{[\varepsilon,k]}}$.

□

The following corollary is then immediate.

Corollary 4.4.14. *Under the assumptions of Lemma 4.4.13, if $\boldsymbol{\lambda} \ll_{\mathbf{m}^{[\varepsilon,I]}} \boldsymbol{\mu}$, then $\boldsymbol{\lambda} \ll_{\mathbf{m}} \boldsymbol{\mu}$.*

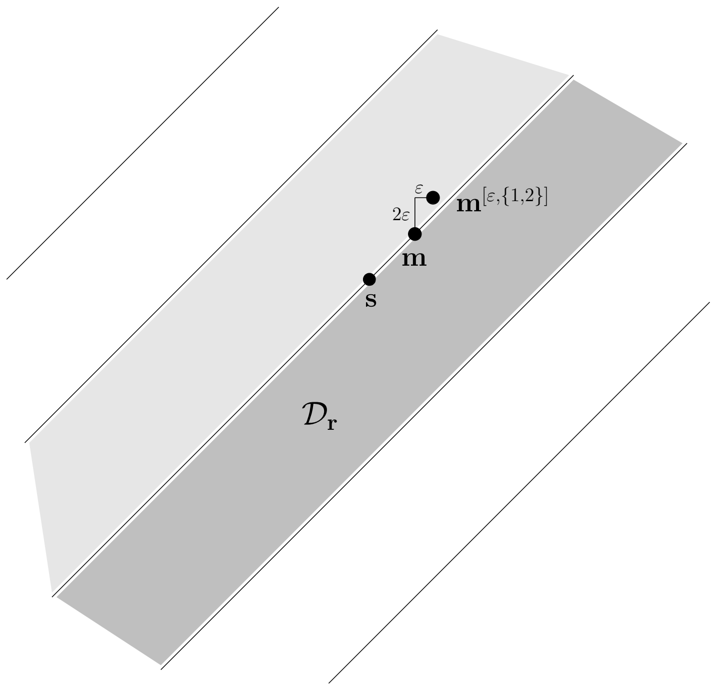


Figure 4.2: The perturbation $\mathbf{m}^{[\varepsilon,\{1,2\}]}$ of $\mathbf{m}$ in level 2.

Proof of Proposition 4.4.11. Recall that we have fixed a weight sequence $\mathbf{m}$ which is asymptotic and belongs to $\mathscr{P}$. Denote then $\mathcal{P}_{i,j}(\mathbf{r})$, for $(i,j) \in J$, the hyperplanes containing $\mathbf{m}$, where $\mathbf{r}$ is the $\mathbf{m}$-adapted multicharge and is asymptotic. Set also $I = \{i, j\ ; (i,j) \in J\}$. Since $\mathbf{r}$ is asymptotic, we have $r_i \neq r_j$ for all $i \neq j$. Let π be the reordering permutation of $\mathbf{r}$, that is, the element of $\mathfrak{S}_l$ verifying $[\pi(i) < \pi(j) \Rightarrow r_i > r_j]$.

Denote also $\delta = \min\limits_{\substack{(i,j)\notin J, j\in I \\ \mathbf{r}'\in\mathcal{C}_e}} d(\mathbf{m}, \mathcal{P}_{i,j}(\mathbf{r}'))$, that is the minimal distance (in the usual sense) between $\mathbf{m}$ and the set of hyperplanes $\mathcal{P}_{i,j}(\mathbf{r}')$ with $j \in I$ but $(i,j) \notin J$ (hence it is positive).

For $M \in \mathrm{Irr}(\mathbf{H}_{k,n})$, write $\mathscr{S}(M) = \{\boldsymbol{\mu} \vdash_l n \,|\, d_{\boldsymbol{\mu},M} \neq 0\}$, as in Definition 2.2.13. Set $\alpha = \min\limits_{\boldsymbol{\mu}\in\mathscr{S}(M)} (\alpha_{\boldsymbol{\lambda},\boldsymbol{\mu}})$, where the elements $\alpha_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ are defined in Lemma 4.4.13, and take $0 < \varepsilon < \min(\alpha, e/l, \delta/l)$.

Consider now the perturbed weight sequence $\mathbf{m}^{[\varepsilon,I]} = (m_1^{[\varepsilon,I]}, \dots, m_l^{[\varepsilon,I]})$. Because $\mathbf{m} \in \mathcal{P}_{i,j}(\mathbf{r})$ for all $(i,j) \in J$, we have

$$m_j - m_i = r_j - r_i. \tag{4.9}$$

Hence, for all $(i,j) \in J$, we have

$$\begin{aligned} m_j^{[\varepsilon,I]} - m_i^{[\varepsilon,I]} &= (m_j + j\varepsilon) - (m_i + i\varepsilon) \\ &= m_j - m_i + (j-i)\varepsilon \\ &= r_j - r_i + (j-i)\varepsilon. \end{aligned}$$

We have $-l < j - i < l$. Because we have chosen $\varepsilon < e/l$, we have $(j-i)\varepsilon < e$. But for $\mathbf{r}', \mathbf{r}'' \in \mathcal{C}_e$, we have $r''_i = r'_i + N_i e$ for some $N_i \in \mathbb{Z}$ and for all $i \in [\![1, l]\!]$. This implies that the weight sequence $\mathbf{m}^{[\varepsilon,I]}$ no longer belongs to any hyperplane of the form $\mathcal{P}_{i,j}(\mathbf{r}')$, with $\mathbf{r}' \in \mathcal{C}_e$ and $i, j \in I$.

Also, since $m_i^{[\varepsilon,I]} = m_i$ for all $i \notin I$ and because of (4.9), we know that $\mathbf{m}^{[\varepsilon,I]}$ does not belong to any hyperplane of the form $\mathcal{P}_{i,j}(\mathbf{r})$ for all $i, j \notin I$ and for all $\mathbf{r} \in \mathcal{C}_s$.

Finally, consider a pair (i, j) with $i \notin I$ and $j \in I$, so that $m_i^{[\varepsilon,I]} = m_i$ and $m_j^{[\varepsilon,I]} = m_j + j\varepsilon$. Then we have

$$\begin{aligned} m_j^{[\varepsilon,I]} - m_i^{[\varepsilon,I]} &= m_j + j\varepsilon - m_i \\ &= m_j - m_i + j\varepsilon \\ &= r_j - r_i + j\varepsilon. \end{aligned}$$

We have $j \leq l$, and since $\varepsilon < \delta/l$, we have $j\varepsilon < \delta$. Therefore, the new weight sequence does not $\mathbf{m}^{[\varepsilon,I]}$ belong to any hyperplane of the form $\mathcal{P}_{i,j}(\mathbf{r}')$ with $j \in I$, $(i, j) \notin J$ and $\mathbf{r}' \in \mathcal{C}_e$.

To sum up, we have just proved that $\mathbf{m}^{[\varepsilon,I]} \notin \mathscr{P}$.

Therefore, by Proposition 4.3.5, $(\mathbf{H}_{k,n}, \mathbf{s})$ admits a canonical basic set with respect to $\ll_{\mathbf{m}^{[\varepsilon,I]}}$, namely a set of twisted Uglov l-partitions, which is equal to the set $\mathcal{K}^{\pi}_{\mathcal{C}_e}(n)$ (see Remark 4.4.8 for instance), where π is the reordering permutation of $\mathbf{r}$. Denote $\mathscr{B} = \mathcal{K}^{\pi}_{\mathcal{C}_e}(n)$.

Since $\mathscr{B}$ is the canonical basic set with respect to $\ll_{\mathbf{m}^{[\varepsilon,I]}}$, there exists a unique $\boldsymbol{\lambda} \in \mathscr{B}$ verifying:

$$\boldsymbol{\lambda} \ll_{\mathbf{m}^{[\varepsilon,I]}} \boldsymbol{\mu} \quad \forall \boldsymbol{\mu} \in \mathscr{S}(M). \tag{4.10}$$

Therefore, by Corollary 4.4.14, we have

$$\boldsymbol{\lambda} \ll_{\mathbf{m}} \boldsymbol{\mu}, \quad \forall \boldsymbol{\mu} \in \mathscr{S}(M).$$

This means that $\mathscr{B}$ is the canonical basic set for $(\mathbf{H}_{k,n}, \mathbf{r})$ with respect to $\ll_{\mathbf{m}}$.

□

4.5 Canonical basic sets for singular weight sequences

Denote by $\mathscr{P}^*$ the set of all $\mathbf{m}$ in $\mathscr{P}$ such that $\mathbf{m}$ is not asymptotic. If $\mathbf{m} \in \mathscr{P}^*$, we say that $\mathbf{m}$ is *singular*. We are going to prove that there is no canonical basic set for $(\mathbf{H}_{k,n}, \mathbf{s})$ with respect to $\ll_{\mathbf{m}}$ in this case.

4.5.1 Non-existence result in the singular case

In the previous section, we have considered perturbations $\mathbf{m}^{[\varepsilon,I]}$ of $\mathbf{m}$. In this section we will need more general perturbations. In fact, for $\rho \in \mathfrak{S}_l$, $I \subset [\![1, l]\!]$ and $\varepsilon > 0$, we define the weight sequence $\mathbf{m}^{[\varepsilon,I,\rho]} = (m_1^{[\varepsilon,I,\rho]}, \dots, m_l^{[\varepsilon,I,\rho]})$ by:

$$m_i^{[\varepsilon,I,\rho]} = \begin{cases} m_i \text{ if } i \notin I \\ m_i + \rho(i)\varepsilon \text{ if } i \in I \end{cases}$$

Remark 4.5.1. Note that, in particular, $\mathbf{m}^{[\varepsilon,I,\mathrm{Id}]} = \mathbf{m}^{[\varepsilon,I]}$.

First of all, it is clear that Lemma 4.4.13 actually holds for all the perturbations $\mathbf{m}^{[\varepsilon,I,\rho]}$ of $\mathbf{m}$. Indeed, as noticed in the very beginning of the proof of 4.4.13, the weight sequences $\mathbf{m}^{[\varepsilon,I,\rho]}$ are also obtained by successive perturbations of the form $\mathbf{m}^{[\varepsilon,k]}$.

In other terms, we have the following property:

Lemma 4.5.2. *Let $\mathbf{m}$ be an arbitrary weight sequence, and take $\boldsymbol{\lambda}, \boldsymbol{\mu} \vdash_l n$, $\boldsymbol{\lambda} \neq \boldsymbol{\mu}$. Let $\alpha_{\boldsymbol{\lambda},\boldsymbol{\mu}} > 0$ be as in Lemma 4.4.13. Then, for all $\varepsilon \in]0, \alpha_{\boldsymbol{\lambda},\boldsymbol{\mu}}[$,*

1.

$$\boldsymbol{\lambda} \ll_{\mathbf{m}} \boldsymbol{\mu} \quad \Rightarrow \quad \left[\begin{array}{c} \forall I \subset [\![1,l]\!], \textit{ either } \quad \boldsymbol{\lambda} \ll_{\mathbf{m}^{[\varepsilon,I,\rho]}} \boldsymbol{\mu} \quad \textit{ or} \\ \boldsymbol{\lambda} \textit{ and } \boldsymbol{\mu} \textit{ are not comparable with respect to } \ll_{\mathbf{m}^{[\varepsilon,I,\rho]}}, \end{array} \right]$$

and

2.

$$\left[\begin{array}{c} \boldsymbol{\lambda} \textit{ and } \boldsymbol{\mu} \textit{ are not comparable} \\ \textit{with respect to } \ll_{\mathbf{m}} \end{array} \right] \Rightarrow \left[\begin{array}{c} \boldsymbol{\lambda} \textit{ and } \boldsymbol{\mu} \textit{ are not comparable with} \\ \textit{respect to } \ll_{\mathbf{m}^{[\varepsilon,I,\rho]}}, \ \forall I \subset [\![1,l]\!]. \end{array} \right].$$

In this section, since $\mathbf{m}$ is singular, the $\mathbf{m}$-adapted multicharge $\mathbf{r}$ is non-asymptotic and $\mathbf{m}$ belongs to $\bigcup_{(i,j)\in J} \mathcal{P}_{i,j}(\mathbf{r})$ for some J. Recall that if we set $I = \{i, j\,; (i,j) \in J\}$ as in Section 4.4.2, we have, for all $(i,j) \in J$

$$m_j^{[\varepsilon,I,\mathrm{Id}]} - m_i^{[\varepsilon,I,\mathrm{Id}]} = s_j - s_i + (j-i)\varepsilon \tag{4.11}$$

One can now consider the perturbations $\mathbf{m}^{[\varepsilon,I,(ij)]}$, for $(i,j) \in J$ (that is, associated to the transposition (ij)). They verify

$$m_j^{[\varepsilon,I,(ij)]} - m_i^{[\varepsilon,I,(ij)]} = r_j - r_i + (i-j)\varepsilon \tag{4.12}$$

Looking at (4.11) and (4.12), we see that $\mathbf{m}^{[\varepsilon,I,\mathrm{Id}]}$ and $\mathbf{m}^{[\varepsilon,I,(ij)]}$ are on opposite sides of $\mathcal{P}_{i,j}(\mathbf{r})$. We are now ready to prove the following proposition.

Proposition 4.5.3. *Let $\mathbf{m}$ be a singular weight sequence. Then $(\mathbf{H}_{k,n}, \mathbf{s})$ does not admit any canonical basic set with respect to $\ll_{\mathbf{m}}$.*

Proof. Suppose that there exists a canonical basic set $\mathscr{B}$ for $(\mathbf{H}_{k,n}, \mathbf{s})$ with respect to $\ll_{\mathbf{m}}$.

For $M \in \mathrm{Irr}(\mathbf{H}_{k,n})$, recall that we have denoted $\mathscr{S}(M) = \{\boldsymbol{\mu} \vdash_l n \,|\, d_{\boldsymbol{\mu},M} \neq 0\}$. By definition, there exists a unique element $\boldsymbol{\lambda}_M \in \mathscr{S}(M)$ such that $\boldsymbol{\lambda}_M \ll_{\mathbf{m}} \boldsymbol{\mu}$ for all $\boldsymbol{\mu} \in \mathscr{S}(M)$.

We follow the same notation as in the proof of Proposition 4.4.11, and take $0 < \varepsilon < \min(\alpha, e/l, \delta/l)$. Then, for the same reason as in that proof, $\mathbf{m}^{[\varepsilon,I,\rho]}$ is regular for all $\rho \in \mathfrak{S}_l$. Hence by Proposition 4.3.5, there exists a canonical basic set $\mathscr{B}^{[\rho]}$ for $(\mathbf{H}_{k,n}, \mathbf{r})$ with respect to $\ll_{\mathbf{m}^{[\varepsilon,I,\rho]}}$, namely the set of some twisted Uglov l-partitions. Since $\mathbf{s}$ is not asymptotic, Remark 4.4.10 implies that there exists ρ_1 and ρ_2 such that

$$\mathscr{B}^{[\rho_1]} \neq \mathscr{B}^{[\rho_2]}. \tag{4.13}$$

Note that this is true for $\rho_1 = \mathrm{Id}$ and $\rho_2 = (ij)$ for some $(i,j) \in J$ because of the remark following (4.11) and (4.12). Since $\mathscr{B}^{[\rho_1]}$ is the canonical basic set with respect to $\ll_{\mathbf{m}^{[\varepsilon,I,\rho_1]}}$, there exists a unique element $\boldsymbol{\lambda}_M^{[1]}$ such that for all $\boldsymbol{\mu} \in \mathscr{S}(M)$, $\boldsymbol{\lambda}_M^{[1]} \ll_{\mathbf{m}^{[\varepsilon,I,\rho_1]}} \boldsymbol{\mu}$. Similarly, there exists a unique element $\boldsymbol{\lambda}_M^{[2]}$ such that for all $\boldsymbol{\mu} \in \mathscr{S}(M)$, $\boldsymbol{\lambda}_M^{[2]} \ll_{\mathbf{m}^{[\varepsilon,I,\rho_2]}} \boldsymbol{\mu}$.

Now by (4.13), there exists a particular $M_0 \in \mathrm{Irr}(\mathbf{H}_{k,n})$ such that

$$\boldsymbol{\lambda}_{M_0}^{[1]} \neq \boldsymbol{\lambda}_{M_0}^{[2]}. \tag{4.14}$$

Thus, we have:

- $\boldsymbol{\lambda}_{M_0}^{[1]} \ll_{\mathbf{m}^{[\varepsilon,I,\rho_1]}} \boldsymbol{\lambda}_{M_0}$ and $\boldsymbol{\lambda}_{M_0} \ll_{\mathbf{m}} \boldsymbol{\lambda}_{M_0}^{[1]}$. But by Lemma 4.5.2 (which applies since $\varepsilon < \alpha$), this not possible if $\boldsymbol{\lambda}_{M_0} \neq \boldsymbol{\lambda}_{M_0}^{[1]}$. Hence $\boldsymbol{\lambda}_{M_0} = \boldsymbol{\lambda}_{M_0}^{[1]}$.

- $\boldsymbol{\lambda}_{M_0}^{[2]} \ll_{\mathbf{m}^{[\varepsilon,I,\rho_2]}} \boldsymbol{\lambda}_{M_0}$ and $\boldsymbol{\lambda}_{M_0} \ll_{\mathbf{m}} \boldsymbol{\lambda}_{M_0}^{[2]}$. Again, by Lemma 4.5.2, this not possible if $\boldsymbol{\lambda}_{M_0} \neq \boldsymbol{\lambda}_{M_0}^{[2]}$. Hence $\boldsymbol{\lambda}_{M_0} = \boldsymbol{\lambda}_{M_0}^{[2]}$.

Hence, $\boldsymbol{\lambda}_{M_0}^{[1]} = \boldsymbol{\lambda}_{M_0}^{[2]}$, which contradicts (4.14). □

As previously mentioned in Section 4.1, a singular weight sequence $\mathbf{m}$ can however yield a canonical basic set for $(\mathbf{H}_{k,n}, \mathbf{r})$, but with $\mathbf{r} \in \mathcal{C} \backslash \mathcal{C}_e$ (i.e. with a different parametrisation of the rows of D). This is illustrated in the following example.

Example 4.5.4. Let $l = 2$, $e = 3$, $n \geq 4$, $\mathbf{s} = (1,0)$. In particular $\mathbf{s}$ is not asymptotic, which one can check directly by computing $\Phi_{\mathbf{s}}(n)$ and $\mathcal{K}_{\mathcal{C}_e}(n)$. Take $\mathbf{r} = \mathbf{s}^\sigma = (0,1)$ (where $\sigma = (12)$), and $m = (0,-1)$. Then $\mathbf{m} \in \mathcal{P}_{1,2}(\mathbf{s})$ since $\mathbf{s} - \mathbf{m} = (1,1)$, and by Proposition 4.5.3, $(\mathbf{H}_{k,n}, \mathbf{s})$ does not admit any canonical basic set with respect to $\ll_{\mathbf{m}}$. However, $\mathbf{r} - \mathbf{m} = (0,2)$, so that $\mathbf{m} \in \mathcal{D}_{\mathbf{r}}$. By Proposition 4.3.1, $(\mathbf{H}_{k,n}, \mathbf{r})$ admits a canonical basic set with respect to $\ll_{\mathbf{m}}$, namely the set $\Phi_{\mathbf{r}}(n)$.

Remark 4.5.5. In the particular case where $l = 2$ and $e = \infty$ (i.e. when η is not a root of unity, cf. Section 2.2.2), one can use a simpler argument to show that there is no canonical basic set. First, note that in this case, $\mathcal{C}_e = \{\mathbf{s}\}$, and $\mathscr{P}$ consists of just the line passing through $\mathbf{s}$ with slope one. Also, $\mathscr{P}^* = \mathscr{P}$. There is a "natural" symbol which encodes the weight of a multipartition $\boldsymbol{\lambda}$ seen as an element of the Fock space

$\mathcal{F}_{\mathbf{s}}$. Because $\mathbf{m}$ is singular, this information is precisely the data carried by the shifted $\mathbf{m}$-symbol of $\boldsymbol{\lambda}$. Now since $e = \infty$, one can then show that the decomposition numbers $d_{\boldsymbol{\mu},\boldsymbol{\lambda}}$ are non-zero only if $\boldsymbol{\mu}$ and $\boldsymbol{\lambda}$ are not comparable with respect to $\ll_{\mathbf{m}}$, which proves that there cannot be a basic set with respect to $\ll_{\mathbf{m}}$.

Note also that explicit formulas are known for computing the elements of the canonical basis of the module $V(\mathbf{s})$ in this case, see [98], which directly shows that all the elements appearing in the decomposition of $G_\infty(\boldsymbol{\lambda}, \mathbf{s})$ have the same symbol up to a permutation of their elements.

4.5.2 Classification theorem

Recall that we have defined in the beginning of this chapter the set

$$\mathscr{P} = \Big\{ m \in \mathbb{Q}^l \,|\, \exists\, i \neq j \text{ such that } (s_i - m_i) - (s_j - m_j) \in e\mathbb{Z} \Big\},$$

and that if $\mathbf{m} \notin \mathscr{P}$, it is called regular. Moreover, when the difference between the components of $\mathbf{s}$ is large, then the Uglov l-partitions stabilise to some twisted Kleshchev l-partitions. A weight sequence with such an adapted l-charge is said to be asymptotic. Finally, if $\mathbf{m}$ is neither regular nor asymptotic, it is called singular.

Putting together Propositions 4.3.5, 4.4.11 and 4.5.3, we have proved:

Theorem 4.5.6. *Given a multicharge $\mathbf{s} \in \mathbb{Z}^l$ and a weight sequence $\mathbf{m} \in \mathbb{Q}^l$, we have the following exhaustive classification:*

- *If $\mathbf{m}$ is regular, then $(\mathbf{H}_{k,n}, \mathbf{s})$ admits a canonical basic set with respect to $\ll_{\mathbf{m}}$, namely the set of σ-twisted Uglov l-partitions $\sigma(\Phi_{\mathbf{r}^{\sigma^{-1}}}(n))$ where σ is the $\mathbf{m}$-adapted permutation and $\mathbf{r}$ is the $\mathbf{m}$-adapted multicharge (cf. Proposition 4.3.5).*

- *If $\mathbf{m}$ is asymptotic, then $(\mathbf{H}_{k,n}, \mathbf{s})$ admits a canonical basic set with respect to $\ll_{\mathbf{m}}$, namely the set of π-twisted Kleshchev l-partitions $\mathcal{K}^{\pi}_{\mathcal{C}_e}(n)$, where π is the reordering permutation of the $\mathbf{m}$-adapted multicharge (cf. Corollary 4.4.6).*

- *If $\mathbf{m}$ is singular, then $(\mathbf{H}_{k,n}, \mathbf{s})$ does not admit any canonical basic set with respect to $\ll_{\mathbf{m}}$.*

Remark 4.5.7. Note that a weight sequence $\mathbf{m}$ can be simultaneously regular and asymptotic. In this case, one must have $\sigma(\Phi_{\mathbf{r}^{\sigma^{-1}}}(n)) = \mathcal{K}^{\pi}_{\mathcal{C}_e}(n)$, which is precisely what is stated in Remark 4.4.8.

Now that we have fully understood which values of $\mathbf{m}$ yield a canonical basic set for $(\mathbf{H}_{k,n}, \mathbf{s})$ with respect to $\ll_{\mathbf{m}}$, we can state a similar result for the order induced by the $\mathbf{a}$-function. Indeed, by the compatibility property (2.3), if $\mathscr{B}$ is the canonical basic set with respect to $\ll_{\mathbf{m}}$, it is also the canonical basic set with respect to the $\mathbf{a}$-function. Hence, the first two assertions in Theorem 4.5.6 still hold for this order. Further, one can prove using the same perturbation arguments that all of the results in the singular case also hold for this order. This leads to:

Theorem 4.5.8.

- *If* $\mathbf{m}$ *is regular, then* $(\mathbf{H}_{k,n}, \mathbf{s})$ *admits a canonical basic set with respect to the* $\mathbf{a}$*-function, namely the set of* σ*-twisted Uglov* l*-partitions* $\sigma(\Phi_{\mathbf{r}^{\sigma^{-1}}}(n))$ *where* σ *is the* $\mathbf{m}$*-adapted permutation and* $\mathbf{r}$ *is the* $\mathbf{m}$*-adapted multicharge.*

- *If* $\mathbf{m}$ *is asymptotic, then* $(\mathbf{H}_{k,n}, \mathbf{s})$ *admits a canonical basic set with respect to the* $\mathbf{a}$*-function, namely the set of* π*-twisted Kleshchev* l*-partitions* $\mathcal{K}^{\pi}_{\mathcal{C}_e}(n)$*, where* π *is the reordering permutation of the* $\mathbf{m}$*-adapted multicharge.*

- *If* $\mathbf{m}$ *is singular, then* $(\mathbf{H}_{k,n}, \mathbf{s})$ *does not admit any canonical basic set with respect to the* $\mathbf{a}$*-function.*

Chapter 5

Crystal isomorphisms in Fock spaces

For $l \in \mathbb{Z}_{>0}$ and $\mathbf{s} \in \mathbb{Z}^l$, consider the corresponding Fock space $\mathcal{F}_\mathbf{s}$. We have seen in Section 3.2 how one can define a crystal graph $B(\mathcal{F}_\mathbf{s})$ for $\mathcal{F}_\mathbf{s}$. Because of the decomposition of $\mathcal{F}_\mathbf{s}$ as an integrable $\mathcal{U}_q(\widehat{\mathfrak{sl}_e})$-module, the crystal graph $B(\mathcal{F}_\mathbf{s})$ consists of several connected components, each of which is isomorphic to the crystal graph of some $V(\mathbf{r})$ (defined in (3.4)), for some appropriate value of $\mathbf{r}$. The aim of this chapter is to determine this graph isomorphism, which we will call the "canonical crystal isomorphism", see Theorem 5.4.26 and Corollary 5.4.28. It contains the results proved in [56].

5.1 Crystal isomorphisms and equivalent multipartitions

According to Section 3.2.2, the crystal graph of the Fock space $B(\mathcal{F}_\mathbf{s})$ has for vertices all the l-partitions, and arrows defined as in Theorem 3.2.10. It has infinitely many connected components, each of them being parametrised by its highest weight vertex. In other terms,

$$B(\mathcal{F}_\mathbf{s}) = \bigsqcup_{\dot{\boldsymbol{\lambda}}} B(\dot{\boldsymbol{\lambda}}, \mathbf{s}), \tag{5.1}$$

where the union is taken over all highest weight vertices $\dot{\boldsymbol{\lambda}}$, i.e. vertices without any removable node.

We can introduce the notion of crystal isomorphism.

Definition 5.1.1. Let $\boldsymbol{\lambda} \in \mathcal{F}_\mathbf{s}$ and $\boldsymbol{\mu} \in \mathcal{F}_\mathbf{r}$. A $\mathcal{U}'_q(\widehat{\mathfrak{sl}_e})$-*crystal isomorphism* is a map $\phi : B(\dot{\boldsymbol{\lambda}}, \mathbf{s}) \longrightarrow B(\dot{\boldsymbol{\mu}}, \mathbf{r})$ verifying:

1. $\phi(\dot{\boldsymbol{\lambda}}) = \dot{\boldsymbol{\mu}}$,
2. $\phi \circ \tilde{f}_i = \tilde{f}_i \circ \phi$ whenever $\tilde{f}_i$ acts non trivially.

By 2., the image of $B(\dot{\boldsymbol{\lambda}}, \mathbf{s})$ under ϕ is the whole crystal $B(\dot{\boldsymbol{\mu}}, \mathbf{r})$. In fact, this definition just says that ϕ intertwines the graph structures of $B(\dot{\boldsymbol{\lambda}}, \mathbf{s})$ and $B(\dot{\boldsymbol{\mu}}, \mathbf{r})$. Accordingly, a crystal isomorphism is necessarily a bijection, and therefore also commutes with the crystal operators $\tilde{e}_i$.

Definition 5.1.2. Let $|\boldsymbol{\lambda}, \mathbf{s}\rangle \in \mathcal{F}_\mathbf{s}$ and $|\boldsymbol{\mu}, \mathbf{r}\rangle \in \mathcal{F}_\mathbf{r}$. We say that $|\boldsymbol{\lambda}, \mathbf{s}\rangle$ and $|\boldsymbol{\mu}, \mathbf{r}\rangle$ (or simply $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$) are *equivalent* if there is a $\mathcal{U}'_q(\widehat{\mathfrak{sl}_e})$-crystal isomorphism ϕ between $B(\overset{\bullet}{\boldsymbol{\lambda}}, \mathbf{s})$ and $B(\overset{\bullet}{\boldsymbol{\mu}}, \mathbf{r})$ such that $\phi(|\boldsymbol{\lambda}, \mathbf{s}\rangle) = |\boldsymbol{\mu}, \mathbf{r}\rangle$.

Remark 5.1.3. In other terms, $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are equivalent if they appear at the same place in their respective crystal graphs. One checks that it is in fact an equivalence relation.

The isomorphisms of $\mathcal{U}'_q(\widehat{\mathfrak{sl}_e})$-modules $V(\mathbf{s}) \simeq V(\mathbf{r})$ whenever $\mathbf{r} \in \mathcal{C}(\mathbf{s})$ (the orbit of $\mathbf{s}$ under the action of $\widehat{\mathfrak{S}_l}$, see Section 1.3.2) yield isomorphisms of crystal graphs between $B(\mathbf{s})$ and $B(\mathbf{r})$, see Example B.0.9. There exist also other natural crystal isomorphisms.

Proposition-Definition 5.1.4. *Let $\boldsymbol{\lambda} \in \mathcal{F}_\mathbf{s}$. There exists a unique l-charge $\mathbf{r} \in \mathscr{D}_e^l$ and a unique FLOTW l-partition $\boldsymbol{\mu} \in \mathcal{F}_\mathbf{r}$ such that $|\boldsymbol{\lambda}, \mathbf{s}\rangle$ and $|\boldsymbol{\mu}, \mathbf{r}\rangle$ are equivalent. The associated $\mathcal{U}'_q(\widehat{\mathfrak{sl}_e})$-crystal isomorphism $B(\boldsymbol{\lambda}, \mathbf{s}) \longrightarrow B(\boldsymbol{\mu}, \mathbf{r})$ is called the canonical crystal isomorphism.*

Proof. First of all, if $\boldsymbol{\lambda} \in B(\mathbf{s})$ then, by the remark just above Proposition 5.1.4, there is a crystal isomorphism between $B(\mathbf{s})$ and $B(\mathbf{r})$ where $\mathbf{r}$ is the representative of $\mathbf{s}$ in the fundamental domain $\mathscr{D}_e^l$.

Suppose now that $|\boldsymbol{\lambda}, \mathbf{s}\rangle \in \mathcal{F}_\mathbf{s}$ such that $B(\boldsymbol{\lambda}, \mathbf{s}) \neq B(\mathbf{s})$. This means that $\boldsymbol{\lambda}$ (as a vertex in its crystal graph) is not in the connected component whose highest weight vertex is the empty multipartition. Then there is a sequence $(i_1, \dots, i_p)$ such that $\tilde{e}_{i_p} \dots \tilde{e}_{i_1}(\boldsymbol{\lambda}) = \overset{\bullet}{\boldsymbol{\lambda}}$, the highest weight vertex in $B(\boldsymbol{\lambda}, \mathbf{s})$. Write $\mathrm{wt}(\overset{\bullet}{\boldsymbol{\lambda}}) = \sum_{i=0}^{e-1} a_i \Lambda_i + d\delta$ and define $\mathbf{r}$ to be the increasing l-charge containing a_i occurences of i. In particular, $\mathbf{r} \in \mathscr{D}_e^l$. Then we have a natural crystal isomorphism $B(\boldsymbol{\lambda}, \mathbf{s}) \overset{\phi}{\simeq} B(\mathbf{r})$, and therefore there is a FLOTW l-partition $\boldsymbol{\mu} := \phi(\boldsymbol{\lambda}) = \tilde{f}_{i_1} \dots \tilde{f}_{i_p}(\boldsymbol{\emptyset})$ in $B(\mathbf{r})$ equivalent to $\boldsymbol{\lambda}$.

These elements are clearly unique, since $\mathscr{D}_e^l$ is a fundamental domain for the action of $\widehat{\mathfrak{S}_l}$. □

The goal of this paper is to find a direct and purely combinatorial way to determine this canonical crystal isomorphism, without having to determine the sequence of operators leading to the highest weight vertex and taking the reverse path in $B(\boldsymbol{\emptyset}, \mathbf{r})$ as explained in the previous proof. Note that this question has been answered in [79] in the particular case where $\boldsymbol{\lambda}$ is a highest weight multipartition. The canonical crystal isomorphism in this case is the so-called "peeling procedure", and is much easier to describe. Of course, in this case, the canonical isomorphism maps $\boldsymbol{\lambda}$ to the empty l-partition.

5.2 The case $e = \infty$

In this section, we consider the particular (and easier) case where $e = \infty$. Recall from paragraph 3.1.5 that the Fock space is also a $\mathcal{U}_q(\mathfrak{sl}_\infty)$-module. In this setting, the notion of being FLOTW for a multipartition simply translates to its symbol being semistandard, as already mentioned in Remark 4.2.2. And as a matter of fact, we know a $\mathcal{U}_q(\mathfrak{sl}_\infty)$-crystal isomorphism which associates to each multipartition a new multipartition whose symbol is semistandard. This is the point of the following section.

5.2.1 Schensted's bumping algorithm and solution of the problem

Let $\boldsymbol{\lambda} \in \mathcal{F}_{\mathbf{s}}$.

We first introduce the reading of the symbol $\mathfrak{B}_{\mathbf{s}}(\boldsymbol{\lambda})$. It is the word obtained by writing the elements of $\mathfrak{B}_{\mathbf{s}}(\boldsymbol{\lambda})$ from right to left, starting from the top row. Denote it by $\mathrm{read}(\boldsymbol{\lambda}, \mathbf{s})$. The Schensted insertion procedure (or bumping procedure) enables to construct a semistandard symbol starting from such a word. We only recall it on an example (5.2.3) below, see also Example 5.5.1 in Section 5.5. For proper background, the reader can refer to e.g. [42] or [108]. Denote by $\mathscr{P}(\mathrm{read}(\boldsymbol{\lambda}, \mathbf{s}))$ the semistandard symbol obtained from $\mathrm{read}(\boldsymbol{\lambda}, \mathbf{s})$ applying this insertion procedure. Finally, we set $\mathbf{RS}(\mathbf{s})$ and $\mathbf{RS}(\boldsymbol{\lambda})$ to be the FLOTW multicharge and multipartition determined by $\mathfrak{B}_{\mathbf{RS}(\mathbf{s})}(\mathbf{RS}(\boldsymbol{\lambda})) = \mathscr{P}(\mathrm{read}(\boldsymbol{\lambda}, \mathbf{s}))$

We further write $\mathbf{RS}$ for the map

$$\begin{array}{rccc} \mathbf{RS}: & B(\overset{\bullet}{\boldsymbol{\lambda}}, \mathbf{s}) & \longrightarrow & B(\overset{\bullet}{\mathbf{RS}}(\boldsymbol{\lambda}), \mathbf{RS}(\mathbf{s})) \\ & |\boldsymbol{\lambda}, \mathbf{s}\rangle & \longmapsto & |\mathbf{RS}(\boldsymbol{\lambda}), \mathbf{RS}(\mathbf{s})\rangle. \end{array}$$

Theorem 5.2.1. *$|\boldsymbol{\lambda}, \mathbf{s}\rangle$ and $|\boldsymbol{\mu}, \mathbf{r}\rangle$ are equivalent if and only if* $\mathscr{P}(\mathrm{read}(\boldsymbol{\lambda}, \mathbf{s})) = \mathscr{P}(\mathrm{read}(\boldsymbol{\mu}, \mathbf{r}))$.

For a proof of this statement, see for instance [96, Section 3] or [101], which state the result for $\mathcal{U}_q(\mathfrak{sl}_e)$-crystals, relying on the original arguments of Kashiwara in [86] and [90]. Moreover, since the symbol associated to $|\mathbf{RS}(\boldsymbol{\lambda}), \mathbf{RS}(\mathbf{s})\rangle$ is semistandard, we have the following result.

Corollary 5.2.2. ***RS*** *is the canonical $\mathcal{U}_q(\mathfrak{sl}_\infty)$-crystal isomorphism.*

Example 5.2.3. $\mathbf{s} = (0, 2, -1)$ and $\boldsymbol{\lambda} = (2.1, 3, 4.1^2)$.

Then $\mathfrak{B}_{\mathbf{s}}(\boldsymbol{\lambda}) = \begin{pmatrix} 0 & 2 & 3 & 7 & & & \\ 0 & 1 & 2 & 3 & 4 & 5 & 9 \\ 0 & 1 & 2 & 4 & 6 & & \end{pmatrix}$.

The associated reading is $\mathrm{read}(\boldsymbol{\lambda}) = 7320954321064210$, and the Schensted algorithm yields

$$\mathscr{P}(\mathrm{read}(\boldsymbol{\lambda})) = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 7 \\ 0 & 1 & 2 & 3 & 6 & 9 & \\ 0 & 2 & 4 & & & & \end{pmatrix}.$$

Hence $= \mathbf{RS}(\mathbf{s}) = (-2, 1, 2)$ and $= \mathbf{RS}(\boldsymbol{\lambda}) = (2.1, 4.2, 1)$.

Property 5.2.4. *Suppose* $\mathbf{s}$ *is such that* $s_1 \leq \cdots \leq s_l$. *Take* $\boldsymbol{\lambda} \in \mathcal{F}_{\mathbf{s}}$. *Then* $|\boldsymbol{RS}(\boldsymbol{\lambda})| \leq |\boldsymbol{\lambda}|$. *Moreover,* $|\boldsymbol{RS}(\boldsymbol{\lambda})| = |\boldsymbol{\lambda}|$ *if and only if* $\boldsymbol{RS}(\boldsymbol{\lambda}) = \boldsymbol{\lambda}$.

Proof. Because of Corollary 5.2.2, we know in particular that $\mathbf{RS}(\boldsymbol{\lambda})$ is in the connected component of $B(\mathcal{F}_{\mathbf{RS(s)}})$ whose highest weight vertex is $\boldsymbol{\emptyset}$. Hence, if we write $\boldsymbol{\emptyset} = \tilde{e}_{i_m} \dots \tilde{e}_{i_1}(\mathbf{RS}(\boldsymbol{\lambda}))$, we have $|\mathbf{RS}(\boldsymbol{\lambda})| = m$. Now, because $\mathbf{RS}$ is a crystal isomorphism, we have $\overset{\bullet}{\boldsymbol{\lambda}} = \tilde{e}_{i_m} \dots \tilde{e}_{i_1}(\boldsymbol{\lambda})$. If $\boldsymbol{\lambda} \neq \mathbf{RS}(\boldsymbol{\lambda})$, then the symbol of $|\boldsymbol{\lambda}, \mathbf{s}\rangle$ is not semistandard, and the fact that $s_1 \leq \cdots \leq s_l$ ensures that $\overset{\bullet}{\boldsymbol{\lambda}} \neq \boldsymbol{\emptyset}$. Therefore, we have $|\boldsymbol{\lambda}| = |\overset{\bullet}{\boldsymbol{\lambda}}| + m > m = |\mathbf{RS}(\boldsymbol{\lambda})|$. $\square$

5.2.2 Another $\mathcal{U}_q(\mathfrak{sl}_\infty)$-crystal isomorphism

Let $\sigma \in \mathfrak{S}_l$, and for $\mathbf{s} = (s_1, \dots, s_l) \in \mathbb{Z}^l$ denote $\mathbf{s}^\sigma = (s_{\sigma(1)}, \dots, s_{\sigma(l)})$.

According to [78, Section 2], we know an explicit combinatorial description of the following $\mathcal{U}_q(\mathfrak{sl}_\infty)$-crystal isomorphism:

$$\begin{array}{rccc} \chi_\sigma : & B(\mathbf{s}) & \longrightarrow & B(\mathbf{s}^\sigma) \\ & \boldsymbol{\lambda} & \longmapsto & \chi_\sigma(\boldsymbol{\lambda}) \end{array}$$

In fact, in [78, Corollary 2.3.3], the map χ_σ is described in the case where σ is a transposition $(c, c+1)$. We do not recall here the combinatorial construction of $\chi_\sigma(\boldsymbol{\lambda})$, since it is not really important for our purpose. However, we notice the following property. It will be used in the proof that the algorithm we construct in Section 5.3 terminates.

Property 5.2.5. *For all* $\sigma \in \mathfrak{S}_l$, *and for all* $\boldsymbol{\lambda} \in B(\mathbf{s})$,

$$|\chi_\sigma(\boldsymbol{\lambda})| = |\boldsymbol{\lambda}|.$$

Proof. It is sufficient to prove it for transpositions $\sigma = (c, c+1)$, and for bipartitions that are highest weight vertices. Recall from [78] that when σ is a transposition $(c, c+1)$, we regard χ_σ as the crystal isomorphism $B(s_c) \otimes B(s_{c+1}) \longrightarrow B(s_{c+1}) \otimes B(s_c)$, and (λ^1, λ^2) as $\lambda^1 \otimes \lambda^2$. Because of Theorem 3.1.15, if the vector $\lambda^1 \otimes \lambda^2$ in $B(s_i) \otimes B(s_j)$ is a highest weight vertex, then λ^1 is a highest weight vertex in $B(s_i)$, hence $\lambda^1 = \emptyset$. Therefore, It is sufficient to check that the property holds for bipartitions of the form $(\emptyset, \lambda)$. Because χ_σ is a crystal isomorphism, $\chi_\sigma((\emptyset, \lambda))$ is a highest weight vertex in $B(s_{c+1}) \otimes B(s_c)$, and thus writes $(\emptyset, \mu)$. Besides,

$$\mathrm{wt}((\emptyset, \lambda)) = \Lambda_{s_c} + \Lambda_{s_{c+1}} - \sum_{k=1}^{p} \alpha_{i_k}$$

on the one hand, with $p = |\lambda|$, and

$$\mathrm{wt}((\emptyset, \mu)) = \Lambda_{s_{c+1}} + \Lambda_{s_c} - \sum_{k=1}^{p} \alpha_{i_k}$$

on the other hand (since χ_σ is a crystal isomorphism), which gives $p = |\mu|$. Hence $|\lambda| = |\mu|$.

$\square$

We also denote simply by χ the isomorphism corresponding to a permutation σ verifying $s_{\sigma(1)} \leq s_{\sigma(2)} \leq \cdots \leq s_{\sigma(l)}$ (i.e. the reordering of $\mathbf{s}$).

5.3 General case and reduction to cylindric multipartitions

5.3.1 Compatibility between $\mathcal{U}'_q(\widehat{\mathfrak{sl}_e})$-crystals and $\mathcal{U}_q(\mathfrak{sl}_\infty)$-crystals

In this section, we use the subscript or superscript e or ∞ to specify which module structure we are interested in, in particular for the (reduced) i-word, crystal operators, crystal graph.

The aim is to show that any $\mathcal{U}_q(\mathfrak{sl}_\infty)$-crystal isomorphism is also a $\mathcal{U}'_q(\widehat{\mathfrak{sl}_e})$-crystal isomorphism. This comes as a natural consequence of the existence of an embedding of $B_e(\overset{\bullet}{\boldsymbol{\lambda}})$ in $B_\infty(\overset{\bullet}{\boldsymbol{\lambda}})$, as explained in [78, Section 4]. Note that the embedding in our case will just map any $\boldsymbol{\lambda} \in B(\overset{\bullet}{\boldsymbol{\lambda}}, \mathbf{s})$ onto itself, unlike in [78].

Lemma 5.3.1. *Let $i \in [\![0, e-1]\!]$. Suppose there is an arrow $\boldsymbol{\lambda} \overset{i}{\longrightarrow} \boldsymbol{\mu}$ in the crystal graph $B_e(\boldsymbol{\lambda})$, and denote $\gamma := [\boldsymbol{\mu}] \backslash [\boldsymbol{\lambda}]$. Then there is an arrow $\boldsymbol{\lambda} \overset{j}{\longrightarrow} \boldsymbol{\mu}$ in the crystal graph $B_\infty(\boldsymbol{\lambda})$, where $j = \mathrm{cont}(\gamma)$.*

Proof. Denote by $w_i^e(\boldsymbol{\lambda})$ (resp. $w_i^\infty(\boldsymbol{\lambda})$) the i-word for $\boldsymbol{\lambda}$ with respect to the $\mathcal{U}'_q(\widehat{\mathfrak{sl}_e})$-crystal (resp. $\mathcal{U}_q(\mathfrak{sl}_\infty)$-crystal) structure. Then $w_i^e(\boldsymbol{\lambda})$ is the concatenation of i-words for the $\mathcal{U}_q(\mathfrak{sl}_\infty)$-crystal structure, precisely

$$w_i^e(\boldsymbol{\lambda}) = \prod_{k \in \mathbb{Z}} w_{i+ke}^\infty(\boldsymbol{\lambda}). \tag{5.2}$$

We further denote $\hat{w}_i^e(\boldsymbol{\lambda})$ and $\hat{w}_i^\infty(\boldsymbol{\lambda})$ the reduced i-words (that is, after recursive cancellation of the factors RA). The node γ is encoded in both $w_i^e(\boldsymbol{\lambda})$ and $w_j^\infty(\boldsymbol{\lambda})$ by a letter A. Now if this letter A does not appear in $\hat{w}_j^\infty(\boldsymbol{\lambda})$, this means that there is a letter R in $w_j^\infty(\boldsymbol{\lambda})$ which simplifies with this A. Hence, because of (5.2), this letter R also appears in $w_i^e(\boldsymbol{\lambda})$ and simplifies with the A encoding γ, and γ cannot be the good addable i-node of $\boldsymbol{\lambda}$ for the $\mathcal{U}'_q(\widehat{\mathfrak{sl}_e})$-crystal structure, whence a contradiction. Thus γ produces a letter A in $\hat{w}_j^\infty(\boldsymbol{\lambda})$.

In fact, this letter A is the rightmost one in $\hat{w}_j^\infty(\boldsymbol{\lambda})$. Indeed, suppose there is another letter A in $\hat{w}_j^\infty(\boldsymbol{\lambda})$ to the right of the A encoding γ. Then it also appears in $\hat{w}_i^e(\boldsymbol{\lambda})$ at the same place (again because of Relation (5.2)). This contradicts the fact that γ is the good addable i-node for the $\mathcal{U}'_q(\widehat{\mathfrak{sl}_e})$-crystal structure.

Therefore, γ is the good addable j-node of $\boldsymbol{\lambda}$ for the $\mathcal{U}_q(\mathfrak{sl}_\infty)$-crystal structure.

□

Lemma 5.3.2. *Let $i \in [\![0, e-1]\!]$, and let φ be a $\mathcal{U}_q(\mathfrak{sl}_\infty)$-crystal isomorphism between connected components. Suppose there is an arrow $\boldsymbol{\lambda} \overset{i}{\longrightarrow} \boldsymbol{\mu}$ in the crystal graph $B_e(\boldsymbol{\lambda})$, and denote $\gamma := [\boldsymbol{\mu}] \backslash [\boldsymbol{\lambda}]$. Then*

1. *there is an arrow* $\varphi(\boldsymbol{\lambda}) \xrightarrow{i} \boldsymbol{\nu}$ *in the crystal graph* $B_e(\varphi(\boldsymbol{\lambda}))$,

2. $\boldsymbol{\nu} = \tilde{f}_j^\infty(\varphi(\boldsymbol{\lambda}))$, *where* $j = \mathrm{cont}(\gamma)$.

Proof. First, for all k, we have the following relation:

$$\hat{w}_k^\infty(\boldsymbol{\lambda}) = \hat{w}_k^\infty(\varphi(\boldsymbol{\lambda})). \tag{5.3}$$

Indeed, if $\hat{w}_k^\infty(\boldsymbol{\lambda}) = A^\alpha R^\beta$, then α can be seen as the number of consecutive arrows labeled by k in $B_\infty(\boldsymbol{\lambda})$ starting from $\boldsymbol{\lambda}$, and β as the number of consecutive arrows labeled by k leading to $\boldsymbol{\lambda}$. Subsequently, the integers α and β are invariant by φ, and the relation (5.3) is verified. Hence, by concatenating, we get

$$\prod_{k\in\mathbb{Z}} \hat{w}_{i+ke}^\infty(\boldsymbol{\lambda}) = \prod_{k\in\mathbb{Z}} \hat{w}_{i+ke}^\infty(\varphi(\boldsymbol{\lambda})), \tag{5.4}$$

and therefore

$$\hat{w}_i^e(\boldsymbol{\lambda}) = \hat{w}_i^e(\varphi(\boldsymbol{\lambda})), \tag{5.5}$$

which proves the first point.

Besides, we know by Lemma 5.3.1 that $\tilde{f}_i^e$ acts like $\tilde{f}_j^\infty$ on $\boldsymbol{\lambda}$. Together with (5.4), this implies that $\tilde{f}_i^e$ acts like $\tilde{f}_j^\infty$ on $\varphi(\boldsymbol{\lambda})$. In other terms, $\boldsymbol{\nu} = \tilde{f}_j^\infty(\varphi(\boldsymbol{\lambda}))$, and the second point is proved. $\square$

Proposition 5.3.3. *Every* $\mathcal{U}_q(\mathfrak{sl}_\infty)$*-crystal isomorphism is also a* $\mathcal{U}'_q(\widehat{\mathfrak{sl}_e})$*-crystal isomorphism.*

Proof. The fact that φ is a $\mathcal{U}_q(\mathfrak{sl}_\infty)$-crystal isomorphism is encoded in the following diagram:

$$\begin{tikzcd}
\boldsymbol{\lambda} \arrow[r, "\varphi"] \arrow[d, "\tilde{f}_j^\infty"'] & \varphi(\boldsymbol{\lambda}) \arrow[d, "\tilde{f}_j^\infty"] \\
\boldsymbol{\mu} \arrow[r, "\varphi"] & \varphi(\boldsymbol{\mu})
\end{tikzcd}$$

The first point of Lemma 5.3.2 tells us that we have:

$$\begin{tikzcd}
\boldsymbol{\lambda} \arrow[r, "\varphi"] \arrow[d, "\tilde{f}_i^e"'] & \varphi(\boldsymbol{\lambda}) \arrow[d, "\tilde{f}_i^e"] \\
\boldsymbol{\mu} & \boldsymbol{\nu}
\end{tikzcd}$$

Besides,

$$\begin{array}{rcll}
\boldsymbol{\nu} & = & \tilde{f}_j^\infty(\varphi(\boldsymbol{\lambda})) & \text{by Point 2 of Lemma 5.3.2} \\
 & = & \varphi(\tilde{f}_j^\infty(\boldsymbol{\lambda})) & \text{because } \varphi \text{ is a } \mathcal{U}_q(\mathfrak{sl}_\infty)\text{-crystal isomorphism} \\
 & = & \varphi(\boldsymbol{\mu}). &
\end{array}$$

Hence we can complete the previous diagram in

$$\begin{tikzcd}
\boldsymbol{\lambda} \arrow[r, "\varphi"] \arrow[d, "\tilde{f}_i^e"'] & \varphi(\boldsymbol{\lambda}) \arrow[d, "\tilde{f}_i^e"] \\
\boldsymbol{\mu} \arrow[r, "\varphi"] & \varphi(\boldsymbol{\mu})
\end{tikzcd}$$

which illustrates the commutation between $\tilde{f}_i^e$ and φ. □

As a consequence, the two particular $\mathcal{U}_q(\mathfrak{sl}_\infty)$-crystal isomorphisms $\mathbf{RS}$ and χ_σ, defined respectively in Section 5.2.1 and 5.2.2, are $\mathcal{U}_q'(\widehat{\mathfrak{sl}_e})$-crystal isomorphisms (for all values of $e \in \mathbb{Z}_{>1}$).

5.3.2 The cyclage isomorphism

One of the most natural $\mathcal{U}_q'(\widehat{\mathfrak{sl}_e})$-crystal isomorphisms to determine is the cyclage isomorphism. Recall that we have already defined the cyclage of a charged multipartition in Chapter 1, Definition 1.4.3.

For $\mathbf{s} = (s_1, \dots, s_l)$, let $\mathbf{s}' = \xi(\mathbf{s}) = (s_l - e, s_1, \dots, s_{l-1})$. Then the following result is easy to show (see for instance [78, Proposition 5.2.1], or [77, Proposition 3.1] for the simpler case $l = 2$):

Proposition 5.3.4. *The map*

$$\begin{array}{cccc} \xi : & B(\overset{\bullet}{\boldsymbol{\lambda}}, \mathbf{s}) & \longrightarrow & B(\mathcal{F}_{\mathbf{s}'}) \\ & (\lambda^1, \dots, \lambda^l) & \longmapsto & (\lambda^l, \lambda^1, \dots, \lambda^{l-1}) \end{array}$$

is a $\mathcal{U}_q'(\widehat{\mathfrak{sl}_e})$-crystal isomorphism. It is called the cyclage isomorphism.

Remark 5.3.5. Note that this was already claimed for Uglov multipartitions in Chapter 4 (discussion page 90), see Remark 4.3.6.

Remark 5.3.6. Recall from Proposition 1.4.4 that the cylindricity condition defined in Definition 1.4.1 is conveniently expressible in terms of symbols using the cyclage ξ.

Remark 5.3.7. We will see (Remark 5.4.23) that the canonical isomorphism we aim to determine can be naturally regarded as a generalisation of this cyclage isomorphism...

Finally, it is straightforward from the definition of ξ that the following property holds:

Property 5.3.8. *For all $\boldsymbol{\lambda} \in \mathcal{F}_{\mathbf{s}}$, we have $|\xi(\boldsymbol{\lambda})| = |\boldsymbol{\lambda}|$.*

5.3.3 Finding a cylindric equivalent multipartition

In this section, we make use of the $\mathcal{U}'_q(\widehat{\mathfrak{sl}_e})$-crystal isomorphisms $\mathbf{RS}$ (see Section 5.2.1) and ξ (defined in Proposition 5.3.4) to construct an algorithm which associates to any charged multipartition $|\boldsymbol{\lambda}, \mathbf{s}\rangle$ an equivalent charged multipartition $|\boldsymbol{\mu}, \mathbf{r}\rangle$ which is cylindric (see Definition 1.4.1). In the sequel, we will denote by $\mathscr{C}_{\mathbf{s}}$ the subset of $\mathcal{F}_{\mathbf{s}}$ of cylindric l-partitions. In particular, this implies that $\mathbf{s} \in \mathscr{S}_e^l$. First of all, let us explain why restricting ourselves to cylindric multipartitions is relevant.

Proposition 5.3.9. *Let $\mathbf{s} \in \mathscr{S}_e^l$. The set $\mathscr{C}_{\mathbf{s}}$ is stable under the action of the crystal operators.*

Proof. Let $\boldsymbol{\lambda} \in \mathscr{C}_{\mathbf{s}}$. By Remark 5.3.6, we know that $\mathfrak{B}_{\mathbf{s}}(\boldsymbol{\lambda})$ is semistandard and $\mathfrak{B}_{\xi(\mathbf{s})}(\xi(\boldsymbol{\lambda}))$ is semistandard.

It is easy to see that $\mathfrak{B}_{\mathbf{s}}(\tilde{f}_i(\boldsymbol{\lambda}))$ (resp. $\mathfrak{B}_{\mathbf{s}}(\tilde{e}_i(\boldsymbol{\lambda}))$) is still semistandard, whenever $\tilde{f}_i$ (resp. $\tilde{e}_i$) acts non trivially on $\boldsymbol{\lambda}$. Indeed, denote γ the good addable i-node and let $j = \mathrm{cont}(\gamma)$. It is encoded by an entry $j+p$, where p is the size of the symbol, see Section 1.1.3. By definition of being a good node is the leftmost i-node amongst all i-node of content j. Hence, there is no other no entry below the entry $j+p$ encoding γ. Since $\tilde{e}_i$ just turns this $j+p$ into $j+p+1$, the symbol of $\tilde{f}_i(\boldsymbol{\lambda})$ is still semistandard. The similar argument applies to $\tilde{e}_i(\boldsymbol{\lambda})$.

Since the symbol of $\xi(\boldsymbol{\lambda})$ is semistandard, by the same argument as above, we deduce that the symbol of $\tilde{f}_i(\xi(\boldsymbol{\lambda}))$ is still semistandard, i.e. the symbol of $\xi(\tilde{f}_i(\boldsymbol{\lambda}))$ is semistandard, since ξ commutes with $\tilde{f}_i$ (by Proposition 5.3.4). This result also holds for $\xi(\tilde{e}_i(\boldsymbol{\lambda}))$ because ξ also commutes with $\tilde{e}_i$ (see the remark following Definition 5.1.1).

By Remark 5.3.6, this means that $\tilde{f}_i(\boldsymbol{\lambda})$ and $\tilde{e}_i(\boldsymbol{\lambda})$ are both cylindric. □

The algorithm expected can now be stated. Firstly, if $\mathfrak{B}_{\mathbf{s}}(\boldsymbol{\lambda})$ is not semistandard, then we can apply $\mathbf{RS}$ to get a charged multipartition whose symbol is semistandard. Hence we can assume that $\mathfrak{B}_{\mathbf{s}}(\boldsymbol{\lambda})$ is semistandard. In particular, this implies that $s_c \leq s_{c+1}$ for all $c \in [\![1, l-1]\!]$.

Then,

1. If $s_l - s_1 < e$, then:
   (a) if $\mathfrak{B}_{\xi(\mathbf{s})}(\xi(\boldsymbol{\lambda}))$ is semistandard, then $|\boldsymbol{\lambda}, \mathbf{s}\rangle$ is cylindric, hence we stop and take $\boldsymbol{\mu} = \boldsymbol{\lambda}$ and $\mathbf{r} = \mathbf{s}$.
   (b) if $\mathfrak{B}_{\xi(\mathbf{s})}(\xi(\boldsymbol{\lambda}))$ is not semistandard, then put $\boldsymbol{\lambda} \leftarrow \mathbf{RS}(\xi(\boldsymbol{\lambda}))$ and $\mathbf{s} \leftarrow \mathbf{RS}(\xi(\mathbf{s}))$ and start again.
2. If $s_l - s_1 \geq e$, then put $\boldsymbol{\lambda} \leftarrow \mathbf{RS}(\xi(\boldsymbol{\lambda}))$ and $\mathbf{s} \leftarrow \mathbf{RS}(\xi(\mathbf{s}))$ and start again.

Proposition 5.3.10. *The algorithm above terminates.*

Proof. For a multicharge $\mathbf{s} = (s_1, \dots s_l)$, we denote $||\mathbf{s}|| := \sum_{k=2}^{l}(s_k - s_1)$. Hence, if $\mathfrak{B}_{\mathbf{s}}(\boldsymbol{\lambda})$ is semistandard, we have $||\mathbf{s}|| \geq 0$. In particular, at each step in the algorithm, this statistic is always non-negative, since we replace $\mathbf{s}$ by $\mathbf{RS}(\xi(\mathbf{s}))$.

Suppose we are in case 1.(b). Since $s_l - s_1 < e$ and $s_1 \leq s_2 \leq \cdots \leq s_l$, we have $(s_l - e) < s_1 \leq s_2 \leq \cdots \leq s_{l-1}$. In other terms, the multicharge $\xi(\mathbf{s}) = (s_l - e, s_1, \dots, s_{l-1})$ is an increasing sequence. Hence Property 5.2.4 applies, and we have $|\mathbf{RS}(\xi(\boldsymbol{\lambda}))| < |\boldsymbol{\lambda}|$.

Suppose we are in case 2. The first thing to understand is that we get the same multipartition and multicharge applying ξ and $\mathbf{RS}$, or applying ξ, then χ (see Section 5.2.2) and $\mathbf{RS}$. Indeed, χ just reorders the multicharge and gives the associated multipartition, which is a transformation already included in $\mathbf{RS}$, (which gives a multipartition whose symbol is semistandard). Hence, we consider that $\mathbf{RS}(\xi(\boldsymbol{\lambda}))$ is obtained by applying successively ξ, then χ, and finally $\mathbf{RS}$, to $\boldsymbol{\lambda}$. In this procedure, it is possible that $\mathbf{RS}$ acts trivially (i.e. that $\chi(\xi(\boldsymbol{\lambda}))$ is already semistandard). In fact,

- If $\mathbf{RS}$ acts non trivially, then on the one hand $\chi(\xi(\boldsymbol{\lambda}))$ is non semistandard; and on the other hand $\chi(\xi(\mathbf{s}))$ is an increasing sequence (by definition of χ). Thus, we have

$$\begin{array}{rcll} |\mathbf{RS}(\chi(\xi(\boldsymbol{\lambda})))| & < & |\chi(\xi(\boldsymbol{\lambda}))| & \text{applying Property 5.2.4} \\ & = & |\boldsymbol{\lambda}| & \text{by Properties 5.2.5 and 5.3.8 .} \end{array}$$

  Hence in this case, $|\mathbf{RS}(\xi(\boldsymbol{\lambda}))| < |\boldsymbol{\lambda}|$.

- If $\mathbf{RS}$ acts trivially, then this argument no longer applies. However, we have $||\mathbf{RS}(\chi(\xi(\mathbf{s})))|| = ||\chi(\xi(\mathbf{s})))|| < ||\mathbf{s}||$. Indeed, denote $\mathbf{s}' = \chi(\xi(\mathbf{s}))$. Since $s_l - s_1 \geq e$, we have $s_l - e \geq s_1$. This implies that the smallest element of $\xi(\mathbf{s}) = (s_l - e, s_1, \dots, s_{l-1})$ is again s_1, and that $s'_1 = s_1$. Hence

$$\begin{aligned} ||\mathbf{s}'|| &= \sum_{k=2}^{l}(s'_k - s'_1) \\ &= \sum_{k=2}^{l}(s'_k - s_1) \\ &= \sum_{k=2}^{l-1}(s_k - s_1) + (s_l - e) - s_1 \\ &= \sum_{k=2}^{l}(s_k - s_1) - e \\ &= ||\mathbf{s}|| - e \\ &< ||\mathbf{s}|| \end{aligned}$$

  Note also that in this case, $|\mathbf{RS}(\xi(\boldsymbol{\lambda}))| = |\chi(\xi(\boldsymbol{\lambda}))| = |\boldsymbol{\lambda}|$ by Properties 5.2.5 and 5.3.8.

We see that at each step, the rank $|.|$ can never increase. In fact, since it is always non-negative, there is necessarily a finite number of steps at which this statistic decreases. Moreover, when the rank does not increase, then the second statistic $||.||$ decreases. Since it can never be negative (as noted in the beginning of the proof), there is also a finite number of such steps. In conclusion, there is a finite number of steps in the algorithm, which means that it terminates.

$\square$

Remark 5.3.11. This algorithm can also be stated in the simpler following way:

1. If $|\boldsymbol{\lambda}, \mathbf{s}\rangle$ is cylindric, then stop and take $\boldsymbol{\mu} = \boldsymbol{\lambda}$ and $\mathbf{r} = \mathbf{s}$.
2. Else, put $\boldsymbol{\lambda} \leftarrow \mathbf{RS}(\xi(\boldsymbol{\lambda}))$ and $\mathbf{s} \leftarrow \mathbf{RS}(\xi(\mathbf{s}))$ and start again.

In other terms, we have proved that for each charged l-partition $|\boldsymbol{\lambda}, \mathbf{s}\rangle$, there exists $m \in \mathbb{Z}_{\geq 0}$ such that $((\mathbf{RS} \circ \xi)^m \circ \mathbf{RS})\ (|\boldsymbol{\lambda}, \mathbf{s}\rangle)$ is cylindric. This integer m a priori depends on $\boldsymbol{\lambda}$. The following proposition claims that it actually does not depend on $\boldsymbol{\lambda}$, but only on the connected component $B(\overset{\bullet}{\boldsymbol{\lambda}}, \mathbf{s})$.

Proposition 5.3.12. *Let $B(\overset{\bullet}{\boldsymbol{\lambda}}, \mathbf{s})$ be a connected component of $B(\mathcal{F}_{\mathbf{s}})$. Then there exists $m \in \mathbb{Z}_{\geq 0}$ such that for all $\boldsymbol{\lambda} \in B(\overset{\bullet}{\boldsymbol{\lambda}}, \mathbf{s})$,*

- $((\boldsymbol{RS} \circ \xi)^m \circ \boldsymbol{RS})\ (|\boldsymbol{\lambda}, \mathbf{s}\rangle)$ *is cylindric, and*
- $((\boldsymbol{RS} \circ \xi)^{m'} \circ \boldsymbol{RS})\ (|\boldsymbol{\lambda}, \mathbf{s}\rangle)$ *is not cylindric for all $m' < m$.*

Proof. Because of Proposition 5.3.10, we know that for each $\boldsymbol{\lambda} \in B(\overset{\bullet}{\boldsymbol{\lambda}}, \mathbf{s})$, there exists $m(\boldsymbol{\lambda}) \in \mathbb{Z}_{\geq 0}$ verifying this property. Write $m_0 = m(\overset{\bullet}{\boldsymbol{\lambda}})$. Now, take any $\boldsymbol{\lambda} \in B(\overset{\bullet}{\boldsymbol{\lambda}}, \mathbf{s})$, and write $\boldsymbol{\lambda} = \tilde{f}_{i_p} \dots \tilde{f}_{i_1}(\overset{\bullet}{\boldsymbol{\lambda}})$. Because $\mathbf{RS}$ and ξ are crystal isomorphisms, they commute with the crystal operators $\tilde{f}_i$. Hence

$$\begin{aligned}
((\mathbf{RS} \circ \xi)^{m_0} \circ \mathbf{RS})\ (|\boldsymbol{\lambda}, \mathbf{s}\rangle) &= ((\mathbf{RS} \circ \xi)^{m_0} \circ \mathbf{RS})\ (\tilde{f}_{i_p} \dots \tilde{f}_{i_1}(|\overset{\bullet}{\boldsymbol{\lambda}}, \mathbf{s}\rangle)) \\
&= \tilde{f}_{i_p} \dots \tilde{f}_{i_1}(((\mathbf{RS} \circ \xi)^{m_0} \circ \mathbf{RS})\ (|\overset{\bullet}{\boldsymbol{\lambda}}, \mathbf{s}\rangle)) \\
&= \tilde{f}_{i_p} \dots \tilde{f}_{i_1}(|\overset{\bullet}{\boldsymbol{\mu}}, \mathbf{r}\rangle) \quad \text{where } |\overset{\bullet}{\boldsymbol{\mu}}, \mathbf{r}\rangle \text{ is cylindric} \\
&=: |\boldsymbol{\mu}, \mathbf{r}\rangle \quad \text{, which is cylindric because of Proposition 5.3.9 .}
\end{aligned}$$

Moreover, is there exists $m' < m_0$ such that $((\mathbf{RS} \circ \xi)^{m'} \circ \mathbf{RS})\ (|\boldsymbol{\lambda}, \mathbf{s}\rangle)$ is cylindric, then by the same argument $((\mathbf{RS} \circ \xi)^{m'} \circ \mathbf{RS})\ (|\overset{\bullet}{\boldsymbol{\lambda}}, \mathbf{s}\rangle)$ is cylindric, which contradicts the minimality of m_0.

Therefore $m(\boldsymbol{\lambda}) = m_0$. $\square$

Example 5.3.13. Set $e = 4$, $l = 3$, $s = (0, 9, 5)$, and $\boldsymbol{\lambda} = (4.2^2.1^3, 5.2^3.1^4, 7.6.4^2.2^2.1^3) \in \mathcal{F}_{\mathbf{s}}$. Firstly, we see that

$$\mathfrak{B}_{\mathbf{s}}(\boldsymbol{\lambda}) = \begin{pmatrix} 0 & 1 & 3 & 4 & 5 & 7 & 8 & 11 & 12 & 15 & 17 & 18 & & & & \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 9 & 10 & 11 & 12 & 14 & 15 & 16 & 20 \\ 0 & 2 & 3 & 4 & 6 & 7 & 10 & & & & & & & & & \end{pmatrix}$$

is not semistandard. Thus we first compute $\tilde{\boldsymbol{\lambda}} := \mathbf{RS}(\boldsymbol{\lambda})$ and $\tilde{\mathbf{s}} := \mathbf{RS}(\mathbf{s})$. We obtain

$$\mathfrak{B}_{\tilde{\mathbf{s}}}(\tilde{\boldsymbol{\lambda}}) = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 10 & 11 & 12 & 14 & 15 & 16 & 17 & 18 \\ 0 & 1 & 3 & 4 & 5 & 7 & 9 & 11 & 12 & 15 & 20 \\ 0 & 2 & 3 & 4 & 6 & 7 & 10 \end{pmatrix},$$

i.e. $\tilde{\boldsymbol{\lambda}} = (4.2^2.1^3, 10.6.4^2.3.2.1^3, 2^5.1^3)$ and $\tilde{\mathbf{s}} = (0,4,10)$. We see that $|\tilde{\boldsymbol{\lambda}}, \tilde{\mathbf{s}}\rangle$ is not cylindric. Hence we compute $\boldsymbol{\lambda}^{(1)} := (\mathbf{RS} \circ \xi)(\boldsymbol{\lambda})$ and $\mathbf{s}^{(1)} := (\mathbf{RS} \circ \xi)(\boldsymbol{\lambda})$. We get

$$\mathfrak{B}_{\mathbf{s}^{(1)}}(\boldsymbol{\lambda}^{(1)}) = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 7 & 8 & 9 & 11 & 12 & 13 & 14 & 15 & 20 \\ 0 & 1 & 3 & 4 & 6 & 7 & 10 & 11 & 12 \\ 0 & 2 & 3 & 4 & 6 & 7 & 10 \end{pmatrix},$$

i.e. $\boldsymbol{\lambda}^{(1)} = (4.2^2.1^3, 4^3.2^2.1^2, 6.2^5.1^3)$ and $s^{(1)} = (0,2,8)$.

We keep on applying $\mathbf{RS} \circ \xi$ until ending up with a cylindric multipartition. In fact, if we denote $\boldsymbol{\lambda}^{(k)} := (\mathbf{RS} \circ \xi)^k(\boldsymbol{\lambda})$ and $\mathbf{s}^{(k)} := (\mathbf{RS} \circ \xi)^k(\mathbf{s})$, we can compute $|\boldsymbol{\lambda}^{(2)}, \mathbf{s}^{(2)}\rangle$, $|\boldsymbol{\lambda}^{(3)}, \mathbf{s}^{(3)}\rangle$, $|\boldsymbol{\lambda}^{(4)}, \mathbf{s}^{(4)}\rangle$, and we finally have

$$\mathfrak{B}_{\mathbf{s}^{(5)}}(\boldsymbol{\lambda}^{(5)}) = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 6 & 7 & 9 & 11 & 12 \\ 0 & 1 & 3 & 4 & 5 & 7 & 8 & 10 & 11 & 16 \\ 0 & 2 & 3 & 4 & 6 & 7 & 10 \end{pmatrix},$$

i.e. $\boldsymbol{\lambda}^{(5)} = (4.2^2.1^3, 7.3^2.2^2.1^3, 3^2.2.1^2)$ and $\mathbf{s}^{(5)} = (-4,-1,-1)$. We see that $|\boldsymbol{\lambda}^{(5)}, \mathbf{s}^{(5)}\rangle$ is cylindric.

This charged multipartition has the following Young diagram with contents:

$$|\boldsymbol{\lambda}^{(5)}, \mathbf{s}^{(5)}\rangle = \left(\begin{ytableau} -4 & -3 & -2 & -1 \\ -5 & -4 \\ -6 & -5 \\ -7 \\ -8 \\ -9 \end{ytableau}, \begin{ytableau} -1 & 0 & 1 & 2 & 3 & 4 & 5 \\ -2 & -1 & 0 \\ -3 & -2 & -1 \\ -4 & -3 \\ -5 & -4 \\ -6 \\ -7 \\ -8 \end{ytableau}, \begin{ytableau} -1 & 0 & 1 \\ -2 & -1 & 0 \\ -3 & -2 \\ -4 \\ -5 \end{ytableau} \right).$$

With this representation, we see that $|\boldsymbol{\lambda}^{(5)}, \mathbf{s}^{(5)}\rangle$ is not FLOTW (cf. Definition 1.4.1). Therefore, It remains to understand how to obtain an equivalent FLOTW multipartition from a cylindric multipartition. This is the point of the next section, which contains the main result of this paper (Theorem 5.4.20).

5.4 The case of cylindric multipartitions

Recall that for $\mathbf{s} \in \mathscr{S}_e^l$, we have denoted $\mathscr{C}_{\mathbf{s}}$ the set of cylindric l-partitions.

5.4.1 Pseudoperiods in a cylindric multipartition

Let $|\boldsymbol{\lambda}, \mathbf{s}\rangle \in \mathscr{C}_{\mathbf{s}}$ such that $\boldsymbol{\lambda}$ is not FLOTW. Then there is a set of parts of the same size, say α, such that the residues at the end of these parts cover $[\![0, e-1]\!]$. This is formalised in the following definition.

Definition 5.4.1.

- The *first pseudoperiod* of $\boldsymbol{\lambda}$ is the sequence $P(\boldsymbol{\lambda})$ of its rightmost nodes

$$\gamma_1 = (a_1, \alpha, c_1), \dots, \gamma_e = (a_e, \alpha, c_e)$$

  verifying:

  1. there is a set of parts of the same size $\alpha \geq 1$ such that the residues at the rightmost nodes of these parts cover $[\![0, e-1]\!]$.
  2. α is the maximal integer verifying 1.
  3. $\mathrm{cont}(\gamma_1) = \max\limits_{\substack{c \in [\![1,l]\!] \\ a \in [\![1, \mathrm{h}(\lambda^c)]\!]}} \mathrm{cont}(a, \alpha, c)$ and $c_1 = \min\limits_{\substack{a \in [\![1, \mathrm{h}(\lambda^c)]\!] \\ \mathrm{cont}(a,\alpha,c) = \mathrm{cont}(\gamma_1)}} c$,
  4. for all $i \in [\![2, e]\!]$, $\mathrm{cont}(\gamma_i) = \max\limits_{\substack{c \in [\![1,l]\!] \\ a \in [\![1, \mathrm{h}(\lambda^c)]\!] \\ \mathrm{cont}(a,\alpha,c) < \mathrm{cont}(\gamma_{i-1})}} \mathrm{cont}(a, \alpha, c)$

     and $c_i = \min\limits_{\substack{a \in [\![1, \mathrm{h}(\lambda^c)]\!] \\ \mathrm{cont}(a,\alpha,c) = \mathrm{cont}(\gamma_i)}} c$.

  In this case, $P(\boldsymbol{\lambda})$ is also called a *α-pseudoperiod* of $\boldsymbol{\lambda}$, and α is called the *width* of $P(\boldsymbol{\lambda})$.

- Denote $\boldsymbol{\lambda}^{[1]} := \boldsymbol{\lambda} \backslash P(\boldsymbol{\lambda})$, that is the multipartition obtained by forgetting [1] in $\boldsymbol{\lambda}$ the parts α whose rightmost node belongs to $P(\boldsymbol{\lambda})$. Let $k \geq 2$. Then the *k-th pseudoperiod of* $\boldsymbol{\lambda}$ is defined recursively as being the first pseudoperiod of $\boldsymbol{\lambda}^{[k]}$, if it exists, where $\boldsymbol{\lambda}^{[k]} := \boldsymbol{\lambda}^{[k-1]} \backslash P(\boldsymbol{\lambda}^{[k-1]})$.

Any k-th pseudoperiod of $\boldsymbol{\lambda}$ is called a *pseudoperiod* of $\boldsymbol{\lambda}$.

Example 5.4.2. Let $e = 3$, $\mathbf{s} = (2, 3, 4)$, and $\boldsymbol{\lambda} = (2.1^2, 2.1^3, 2.1^4)$. One checks that $\boldsymbol{\lambda}$ is cylindric for e but not FLOTW. Then $\boldsymbol{\lambda}$ has the following diagram with contents:

$$\boldsymbol{\lambda} = \left(\begin{array}{|c|c|} \hline 2 & \color{blue}{3} \\ \hline \color{red}{1} & \\ \cline{1-1} \color{green}{0} & \\ \cline{1-1} \end{array} , \begin{array}{|c|c|} \hline 3 & \color{blue}{4} \\ \hline \color{red}{2} & \\ \cline{1-1} \color{green}{1} & \\ \cline{1-1} 0 & \\ \cline{1-1} \end{array} , \begin{array}{|c|c|} \hline 4 & \color{blue}{5} \\ \hline \color{red}{3} & \\ \cline{1-1} \color{green}{2} & \\ \cline{1-1} 1 & \\ \cline{1-1} 0 & \\ \cline{1-1} \end{array} \right).$$

Then $\boldsymbol{\lambda}$ has one 2-pseudoperiod and two 1-pseudoperiods. Its first pseudoperiod consists of $\gamma_1 = (1, 2, 3), \gamma_2 = (1, 2, 2)$ and $\gamma_3 = (1, 2, 1)$, with respective contents 5, 4 and 3, colored in blue. The second pseudoperiod is $(\gamma_1 = (2, 1, 3), \gamma_2 = (2, 1, 2), \gamma_3 = (2, 1, 1))$, with red contents; and the third (and last) pseudoperiod is $(\gamma_1 = (3, 1, 3), \gamma_2 = (3, 1, 2), \gamma_3 = (3, 1, 1))$, with green contents.

Lemma 5.4.3. *1.* $\mathrm{cont}(\gamma_i) = \mathrm{cont}(\gamma_{i-1}) - 1$ *for all* $i \in [\![2, e]\!]$. *In other terms, the contents of the elements of the pseudoperiod are consecutive integers.*

[1]This means that one considers only the nodes of $\boldsymbol{\lambda}$ that are in parts whose rightmost node is not in $P(\boldsymbol{\lambda})$, but without changing the indexation nor the contents of these nodes.

2. $c_i \leq c_{i-1}$ *for all* $i \in [\![2, e]\!]$.

Proof. 1. Suppose there is a gap in the sequence of these contents. Then the pseudoperiod must spread over $e+1$ columns in the symbol $\mathfrak{B}_{\mathbf{s}}(\boldsymbol{\lambda})$. Denote by $\mathfrak{b}$ the integer of $\mathfrak{B}_{\mathbf{s}}(\boldsymbol{\lambda})$ corresponding to the last element of $P(\boldsymbol{\lambda})$, and k the column where it appears. The integer of $\mathfrak{B}_{\mathbf{s}}(\boldsymbol{\lambda})$ corresponding to the gap must be in column $k+1$, and since $\boldsymbol{\lambda}$ is cylindric, it must be greater than or equal to $\mathfrak{b}+e$. In fact, it cannot be greater than $\mathfrak{b}+e$ since the corresponding part is below a part of size α, and it has to correspond to a part of size α, and there cannot be a gap, whence a contradiction.

2. Since the nodes of $P(\boldsymbol{\lambda})$ are the rightmost nodes of parts of the same size α, together with the fact $s_1 \leq \cdots \leq s_l$, and point 1., γ_i is necessarily either to the left of γ_{i-1} or in the same component.

□

Remark 5.4.4. If $\alpha = \max_{i,j} \lambda^i_j$, then the first pseudoperiod corresponds to a "period" in $\mathfrak{B}_{\mathbf{s}}(\boldsymbol{\lambda})$, accordingly to [79, Definition 2.2] This is the case in Example 5.4.2. In the case where each pseudoperiod corresponds to a period in the symbol associated to $\boldsymbol{\lambda}^{[k]}$, one can directly recover the empty l-partition and the corresponding multicharge using the "peeling procedure" explained in [79]. However, in general, $\mathfrak{B}_{\mathbf{s}}(\boldsymbol{\lambda})$ might not have a period, as shown in the following example.

Example 5.4.5. $e = 4$, $\mathbf{s} = (5, 6, 8)$ and $\boldsymbol{\lambda} = (6^2.2.1, 3.2^3.1^2, 6.2^2.1^3)$. Then $\boldsymbol{\lambda} \in \mathscr{C}_{\mathbf{s}}$ but is not FLOTW for e. It has the following Young diagram with residues:

$$\boldsymbol{\lambda} = \left(\begin{array}{|c|c|c|c|c|c|} \hline 5 & 6 & 7 & 8 & 9 & 10 \\ \hline 4 & 5 & 6 & 7 & 8 & 9 \\ \hline 3 & 4 \\ \cline{1-2} 2 \\ \cline{1-1} 1 \\ \cline{1-1} \end{array} \;,\; \begin{array}{|c|c|c|} \hline 6 & 7 & 8 \\ \hline 5 & 6 \\ \cline{1-2} 4 & 5 \\ \cline{1-2} 3 & 4 \\ \cline{1-2} 2 \\ \cline{1-1} 1 \\ \cline{1-1} \end{array} \;,\; \begin{array}{|c|c|c|c|c|c|} \hline 8 & 9 & 10 & 11 & 12 & 13 \\ \hline 7 & 8 \\ \cline{1-2} 6 & 7 \\ \cline{1-2} 5 \\ \cline{1-1} 4 \\ \cline{1-1} 3 \\ \cline{1-1} \end{array} \right)$$

Then there is a 2-pseudoperiod and a 1-pseudoperiod. The first pseudoperiod of $\boldsymbol{\lambda}$ consists of the nodes $\gamma_1 = (2, 2, 3)$, $\gamma_2 = (3, 2, 3)$, $\gamma_3 = (2, 2, 2)$ and $\gamma_4 = (3, 2, 2)$, with respective contents 8, 7, 6 and 5. The 1-pseudoperiod is $((4, 1, 3), (5, 1, 3), (6, 1, 3), (4, 1, 1))$.

Of course, one could also describe pseudoperiods on the $\mathbf{s}$-symbol associated to $\boldsymbol{\lambda}$. However, this approach is not that convenient, and in the setting of cylindric multipartitions, we favour the "Young diagram with contents" approach, which encodes the same information. Nevertheless, we notice this property, which will be used in the proof of Lemma 5.4.17:

Proposition 5.4.6. *Let $P(\boldsymbol{\lambda})$ be a pseudoperiod of $\boldsymbol{\lambda}$. Denote by B the set of entries of $\mathfrak{B}_{\mathbf{s}}(\boldsymbol{\lambda})$ corresponding to the nodes of $P(\boldsymbol{\lambda})$. Then each column of $\mathfrak{B}_{\mathbf{s}}(\boldsymbol{\lambda})$ contains at most one element of B. Moreover, the elements of B appear in consecutive columns of $\mathfrak{B}_{\mathbf{s}}(\boldsymbol{\lambda})$.*

Proof. This is direct from the fact that the nodes of $P(\boldsymbol{\lambda})$ are all rightmost nodes of parts of the same size α, together with Lemma 5.4.3. □

We will now determine the canonical $\mathcal{U}'_q(\widehat{\mathfrak{sl}_e})$-crystal isomorphism for cylindric multipartitions. In the following section, we only determine the suitable multicharge. In Section 5.4.3, we explain how to construct the actual corresponding FLOTW multipartition.

5.4.2 Determining the multicharge

In [79], Jacon and Lecouvey have proved that when $\boldsymbol{\lambda} = \overset{\bullet}{\boldsymbol{\lambda}}$ is a highest vertex, then it suffices to "peel" the symbol of $\boldsymbol{\lambda}$ in order to get an empty equivalent multipartition. We do not recall here this procedure in detail, but it basically consists in removing all periods in the symbol of $\boldsymbol{\lambda}$ (see Example 5.4.10).

When we start from a multipartition $\boldsymbol{\lambda}$ which is no longer a highest weight vertex, we can, in general, no longer peel the symbol, for it does not necessarily contain a period anymore (see Remark 5.4.4). However, the multicharge we look for is constant along the crystal, hence entirely determined by the highest weight vertex $\overset{\bullet}{\boldsymbol{\lambda}}$. Therefore, it is the representative in $\mathscr{D}_e^l$ of the multicharge associated to the empty multipartition obtained after peeling $\overset{\bullet}{\boldsymbol{\lambda}}$. We denote it by $\varphi(\mathbf{s})$.

Now, since $|\overset{\bullet}{\boldsymbol{\lambda}}, \mathbf{s}\rangle$ is already cylindric, it turns out that the period can be easily read in the symbol of $\overset{\bullet}{\boldsymbol{\lambda}}$. In fact, we have the following property:

Proposition 5.4.7. *Let $|\boldsymbol{\lambda}, \mathbf{s}\rangle \in \mathscr{C}_{\mathbf{s}}$ such that $\mathfrak{B}_{\mathbf{s}}(\boldsymbol{\lambda})$ has a period P. Then each of the rightmost e columns of $\mathfrak{B}_{\mathbf{s}}(\boldsymbol{\lambda})$ contains a unique element of P.*

Proof. Since a period is nothing but a pseudoperiod whose width is the largest part in $\boldsymbol{\lambda}$ (cf. Remark 5.4.4), this result is just a particular case of Proposition 5.4.6. □

Hence, deleting the first period (first step of the peeling) $\overset{\bullet}{\boldsymbol{\lambda}}$, we get a multicharge $\mathbf{s}^{(1)}$ verifying

$$\begin{aligned}
s_l^{(1)} &= s_l - (s_l - s_{l-1}) = s_{l-1} \\
s_{l-1}^{(1)} &= s_{l-1} - (s_{l-1} - s_{l-2}) = s_{l-2} \\
&\vdots \\
s_2^{(1)} &= s_2 - (s_2 - s_1) = s_1 \\
s_1^{(1)} &= s_1 - (e - (s_l - s_1)) = s_l - e.
\end{aligned}$$

In other terms, we have $\mathbf{s}^{(1)} = \xi(\mathbf{s})$, where ξ is the cyclage operator defined in Section 5.3.2.

Applying this recursively, we obtain

Proposition 5.4.8. *For all $k \geq 1$, $\mathbf{s}^{(k)} = \xi^k(\mathbf{s})$, where $\mathbf{s}^{(k)}$ denotes the multicharge associated with the peeled symbol after k steps (with $\mathbf{s}^{(0)} = \mathbf{s}$).*

Remark 5.4.9. It is possible that, at some step, a multicharge $\mathbf{s}^{(k)}$ will not be in $\mathscr{S}_e^l$ anymore. However, for any k, one always has $s_i^{(k)} \leq s_j^{(k)}$ for $i < j$, and $s_l^{(k)} - s_1^{(k)} \leq e$.

Moreover, if $s_l^{(k)} - s_1^{(k)} = e$, then $s_l^{(k+p)} - s_1^{(k+p)} < e$, where $p \geq 1$ is the number of components of $s^{(k)}$ equal to $s_l^{(k)}$.

Note that this never happens if $s_i < s_j$ for all $i < j$.

Example 5.4.10. $e = 3$, $\mathbf{s} = (3,3,4)$, $\boldsymbol{\lambda} = (3^2.2, 2^2.1, 3.1^2) = \dot{\boldsymbol{\lambda}}$. One checks that $|\boldsymbol{\lambda}, \mathbf{s}\rangle$ is cylindric but not FLOTW. The associated symbol is

$$\begin{pmatrix} 0 & 1 & 3 & 4 & 7 \\ 0 & 2 & 4 & 5 & \\ 0 & 3 & 5 & 6 & \end{pmatrix}$$

Peeling this symbol, we get successively

$$\begin{pmatrix} 0 & 1 & 3 & 4 \\ 0 & 2 & 4 & 5 \\ 0 & 3 & & \end{pmatrix} \quad \text{and} \quad \mathbf{s}^{(1)} = (1,3,3),$$

$$\begin{pmatrix} 0 & 1 & 3 & 4 \\ 0 & 2 & & \\ 0 & & & \end{pmatrix} \quad \text{and} \quad \mathbf{s}^{(2)} = (0,1,3),$$

$$\begin{pmatrix} 0 & 1 \\ 0 & \\ 0 & \end{pmatrix} \quad \text{and} \quad \mathbf{s}^{(3)} = (0,0,1).$$

Note that $\mathbf{s}^{(2)} \notin \mathscr{S}_e^l$.

The following proposition is now easy to prove.

Proposition 5.4.11. *Let $\mathbf{s} \in \mathscr{S}_e^l$. There exists $k \in \mathbb{Z}$ such that $\xi^k(\mathbf{s}) \in \mathscr{D}_e^l$. In fact, we have $\varphi(\mathbf{s}) = \xi^k(\mathbf{s})$.*

Remark 5.4.12. This says that the peeling procedure is useless when the multicharge is already in $\mathscr{S}_e^l$.

Proof. Recall that $\varphi(\mathbf{s})$ is the representative of the multicharge $\mathbf{s}'$ associated to the peeled symbol of $\dot{\boldsymbol{\lambda}}$. Because of Consequence 5.4.8, $\mathbf{s}'$ is obtained from $\mathbf{s}$ by applying several times (say t times) ξ. In other terms, $\mathbf{s}' = \xi^t(\mathbf{s})$. Now,

- if $\mathbf{s}' \in \mathscr{D}_e^l$, then $\varphi(\mathbf{s}) = \mathbf{s}'$ and $k = t$.
- if $\mathbf{s}' \in \mathscr{S}_e^l \backslash \mathscr{D}_e$, then the representative of $\mathbf{s}'$ in $\mathscr{D}_e^l$ is of the form $\xi^v(\mathbf{s}')$ for some $v \in \mathbb{Z}$. Hence, $\varphi(\mathbf{s}) = \xi^k(\mathbf{s})$ with $k = t + v$.
- if $\mathbf{s}' \notin \mathscr{S}_e^l$, we are however ensured (see Remark 5.4.9) that there exists $p \in [\![1, e-1]\!]$ such that $\xi^p(\mathbf{s}') \in \mathscr{S}_e^l$. We are then in the previous situation, i.e. there exists $v \in \mathbb{Z}$ such that $\xi^{p+v}(\mathbf{s}') \in \mathscr{D}_e^l$, and therefore $\varphi(\mathbf{s}) = \xi^k(\mathbf{s})$ with $k = t + p + v$.

$\square$

Example 5.4.13. As in Example 5.4.10, take $e=3$, $\mathbf{s}=(3,3,4)$ and $\boldsymbol{\lambda}=(3^2.2,2^2.1,3.1^2)$. Then $\xi^3(\mathbf{s})=(0,0,1)\in\mathscr{D}_e^l$. Hence $\varphi(\mathbf{s})=(0,0,1)$.

5.4.3 Determining the FLOTW multipartition

In order to compute the multipartition $\varphi(\boldsymbol{\lambda})$, we need to introduce a new crystal isomorphism, which acts on cylindric multipartitions. In fact, the only difference between a cylindric multipartition and a FLOTW multipartition is the possible presence of pseudoperiods. Therefore, we want to determine an isomorphism which maps a cylindric multipartition to another cylindric multipartition with one less pseudoperiod. Applying this recursively, we will eventually end up with a FLOTW multipartition equivalent to $\boldsymbol{\lambda}$.

Take $\mathbf{s}\in\mathscr{S}_e^l$ and $\boldsymbol{\lambda}\in\mathscr{C}_{\mathbf{s}}$. Let α be the width of $P(\boldsymbol{\lambda})$, the first pseudoperiod of $\boldsymbol{\lambda}$. Denote by $\psi(\boldsymbol{\lambda})$ the multipartition $\boldsymbol{\mu}$ charged by $\xi(\mathbf{s})$ defined as follows:

- $\boldsymbol{\mu}^c$ contains all parts λ_a^c of λ^c such that $1\leq\lambda_a^c<\alpha$, for $c\in[\![1,l]\!]$,
- $\boldsymbol{\mu}^c$ contains all parts λ_a^{c-1} of λ^{c-1} such that $\lambda_a^{c-1}>\alpha$ for $c>1$, and $\boldsymbol{\mu}^1$ contains all parts λ_a^l of λ^l such that $\lambda_a^l>\alpha$,
- $\boldsymbol{\mu}^c$ contains all parts $\lambda_a^c=\alpha$ whose rightmost node does not belong to $P(\boldsymbol{\lambda})$, for $c\in[\![1,l]\!]$.

This naturally defines a mapping $\gamma\longleftrightarrow\Gamma$ from the set of nodes of $\boldsymbol{\lambda}\backslash P(\boldsymbol{\lambda})$ (see Definition 5.4.1) onto the set of nodes of $\psi(\boldsymbol{\lambda})$. We then say that Γ is *canonically associated to* γ, and conversely.

Example 5.4.14. Let us go back to Example 5.4.5. We had $e=4$, $\mathbf{s}=(5,6,8)$ and

$$\boldsymbol{\lambda}=\left(\begin{array}{llllll}5&6&7&8&9&10\\4&5&6&7&8&9\\3&4\\2\\1\end{array},\ \begin{array}{lll}6&7&8\\5&\mathbf{6}\\4&\mathbf{5}\\3&4\\2\\1\end{array},\ \begin{array}{llllll}8&9&10&11&12&13\\7&\mathbf{8}\\6&\mathbf{7}\\5\\4\\3\end{array}\right),$$

where the bold contents correspond to the first pseudoperiod (whose width is $\alpha=2$).

Then $\xi(\mathbf{s})=(4,5,6)$, and

$$\psi(\boldsymbol{\lambda})=\left(\begin{array}{llllll}4&5&6&7&8&9\\3&4\\2\\1\end{array},\ \begin{array}{llllll}5&6&7&8&9&10\\4&5&6&7&8&9\\3&4\\2\\1\end{array},\ \begin{array}{lll}6&7&8\\5\\4\\3\end{array}\right)=(6.2.1^2,6^2.2.1^2,3.1^3)$$

We observe that the contents of canonically associated nodes are unchanged, except for the nodes of $\boldsymbol{\lambda}$ that lie in part of λ^l greater than α, whose content is translated by $-e$. More formally, this writes:

Proposition 5.4.15. *Denote $\psi(\boldsymbol{\lambda}) =: \boldsymbol{\mu}$.*

1. *Let $\Gamma = (A, B, C)$ be a node of $\boldsymbol{\mu}$ with $\mu_A^C > \alpha$. Denote by γ the node of $\boldsymbol{\lambda}$ canonically associated to γ.*
   - *If $c > 1$, then $\mathrm{cont}_{\boldsymbol{\mu}}(\Gamma) = \mathrm{cont}_{\boldsymbol{\lambda}}(\gamma)$.*
   - *If $c = 1$, then $\mathrm{cont}_{\boldsymbol{\mu}}(\Gamma) = \mathrm{cont}_{\boldsymbol{\lambda}}(\gamma) - e$.*
2. *Let $\Gamma = (A, B, C)$ be a node of $\boldsymbol{\mu}$ with $\mu_A^C \leq \alpha$. Denote by γ its canonically associated node in $\boldsymbol{\lambda}$. Then $\mathrm{cont}_{\boldsymbol{\mu}}(\Gamma) = \mathrm{cont}_{\boldsymbol{\lambda}}(\gamma)$.*

Notation 5.4.16. Let $\mathbf{s}$ be a l-charge in $\mathscr{S}_e^l$, $\boldsymbol{\lambda} \in \mathscr{C}_{\mathbf{s}}$ non FLOTW, and $c \in [\![1, l]\!]$. Set α to be the width of the first pseudoperiod of $\boldsymbol{\lambda}$. We denote:

- $N_c^{>\alpha}$ the number of parts greater than α in λ^c,
- N_c^{α} the number of parts equal to α in λ^c that are deleted in $\boldsymbol{\lambda}$ to get $\psi(\boldsymbol{\lambda})$ (i.e. parts whose rightmost node belongs to the pseudoperiod).

Proof. Of course, it is sufficient to prove this for only one node in each part considered, since the contents of all other nodes of the part is then determined. We prove it only for the leftmost nodes, i.e. the ones of the form $(A, 1, C)$.

1. This is clear since the multicharge associated to $\boldsymbol{\mu}$ is simply the cyclage of $\mathbf{s}$ (that shifts $\mathbf{s}$ "to the right" and maps s_1 to $s_l - e$), and the parts greater than α are similarly shifted in $\boldsymbol{\lambda}$ to get $\boldsymbol{\mu}$.
2. Let $\Gamma = (A, 1, C)$ be a node in $\boldsymbol{\mu}$ such that $\mu_A^C \leq \alpha$, so that $\gamma = (a, 1, c)$ with $c = C$.

   First, assume $C > 1$. Then
   $$\mathrm{cont}_{\boldsymbol{\mu}}(\Gamma) = s_{C-1} - N_{C-1}^{>\alpha}.$$
   On the other hand,
   $$\mathrm{cont}_{\boldsymbol{\lambda}}(\gamma) = s_C - N_C^{>\alpha} - N_C^{\alpha}.$$
   Now by definition of $P(\boldsymbol{\lambda})$, which is charged by $\xi(\mathbf{s})$, we have
   $$s_C - s_{C-1} = N_C^{>\alpha} + N_C^{\alpha} - N_{c-1}^{>\alpha},$$
   which is equivalent to
   $$N_C^{\alpha} = s_C - N_C^{>\alpha} - s_{C-1} - N_{C-1}^{>\alpha}.$$
   Hence we have
   $$\begin{array}{rcl} \mathrm{cont}_{\boldsymbol{\lambda}}(\gamma) & = & s_C - N_C^{>\alpha} - N_C^{\alpha} \\ & = & s_C - N_C^{>\alpha} - (s_C - N_C^{>\alpha} - (s_{C-1} - N_{C-1}^{>\alpha})) \\ & = & s_{C-1} - N_{C-1}^{>\alpha} \\ & = & \mathrm{cont}_{\boldsymbol{\mu}}(\Gamma). \end{array}$$

Now, assume $C=1$. The argument is the same:

$$\mathrm{cont}_{\boldsymbol{\mu}}(\Gamma) = s_l - e - N_l^{>\alpha}, \quad \text{and}$$

$$\mathrm{cont}_{\boldsymbol{\lambda}}(\gamma) = s_1 - N_1^{>\alpha} - N_1^{\alpha}.$$

Moreover, we have:

$$\begin{array}{rcll} N_1^{\alpha} & = & e - \sum_{d=2}^{l} N_d^{\alpha} & \\ & = & e - \sum_{d=2}^{l} (r_d - r_{d-1}) & \\ & = & e - r_l + r_1 & \\ & = & e - (s_l - N_l^{>\alpha}) + (s_1 - N_1^{>\alpha}) & \text{using the above case,} \end{array}$$

which implies that

$$\begin{array}{rcl} \mathrm{cont}_{\boldsymbol{\mu}}(\Gamma) & = & s_l - e - N_l^{>\alpha} \\ & = & s_1 - N_1^{>\alpha} - e + s_l - N_l^{>\alpha} - s_1 + N_1^{>\alpha} \\ & = & s_l - e - N_l^{>\alpha} \\ & = & \mathrm{cont}_{\boldsymbol{\lambda}}(\gamma). \end{array}$$

□

We now aim to prove that the map ψ we have just defined is in fact a crystal isomorphism between connected components of Fock spaces crystals (this is upcoming Theorem 5.4.20). In order to do that, we will need the following three lemmas, in which we investigate the compatibility between ψ and the possible actions of the crystal operators $\tilde{f}_i$. For the sake of clarity (the proofs of these statements being rather technical), they are proved in Appendix A.

Lemma 5.4.17. *Suppose that $\gamma^+ = (a, \alpha+1, c)$ is the good addable i-node of $\boldsymbol{\lambda}$, with $\gamma \in P(\boldsymbol{\lambda})$. Then*

- *$\Delta^+ = (a, \alpha+1, c+1)$ is the good addable i-node of $\psi(\boldsymbol{\lambda})$ if $1 \leq c < l$,*
- *$\Delta^+ = (a, \alpha+1, 1)$ is the good addable i-node of $\psi(\boldsymbol{\lambda})$ if $c = l$.*

Lemma 5.4.18. *Suppose that $\gamma^+ = (a, \lambda_a^c + 1, c)$ is the good addable i-node of $\boldsymbol{\lambda}$, with $\lambda_a^c < \alpha$ or [$\lambda_a^c = \alpha$ and $\gamma \notin P(\boldsymbol{\lambda})$]. Then $\Gamma^+ = (a - D, \lambda_a^c + 1, c)$ is the good addable i-node of $\psi(\boldsymbol{\lambda})$, where*

- *$D = N_c^{>\alpha} - N_{c-1}^{>\alpha} + N_c^{\alpha} - N_{c-1}^{\alpha}$ if $c \geq 1$,*
- *$D = N_1^{>\alpha} - N_l^{>\alpha} + N_1^{\alpha} - N_l^{\alpha}$ if $c = 1$ (see Notation 5.4.16).*

Lemma 5.4.19. *Suppose that $\gamma^+ = (a, \lambda_a^c + 1, c)$ is the good addable i-node of $\boldsymbol{\lambda}$, with $\lambda_a^c > \alpha$. Then*

- $\Gamma^+ = (a, \lambda_a^c + 1, c+1)$ *is the good addable i-node of* $\psi(\boldsymbol{\lambda})$ *if* $1 \leq c < l$,
- $\Gamma^+ = (a, \lambda_a^c + 1, 1)$ *is the good addable i-node of* $\psi(\boldsymbol{\lambda})$ *if* $c = l$.

We are now ready to prove the following key result.

Theorem 5.4.20. *Let* $|\overset{\bullet}{\boldsymbol{\lambda}}, \mathbf{s}\rangle$ *be a cylindric l-partition. The map*

$$\begin{array}{cccc} \psi: & B(\overset{\bullet}{\boldsymbol{\lambda}}, \mathbf{s}) & \longrightarrow & B(\psi(\overset{\bullet}{\boldsymbol{\lambda}}), \xi(\mathbf{s})) \\ & \boldsymbol{\lambda} & \longmapsto & \psi(\boldsymbol{\lambda}) \end{array}$$

is a crystal isomorphism. We call it the reduction isomorphism for cylindric multipartitions.

Proof. We need to prove that for all $i \in [\![0, e-1]\!]$,

$$\tilde{f}_i(\psi(\boldsymbol{\lambda})) = \psi(\tilde{f}_i(\boldsymbol{\lambda})). \tag{5.6}$$

Thanks to the previous lemmas, we know precisely what $\tilde{f}_i(\psi(\boldsymbol{\lambda}))$ is. It remains to understand the right hand side of (5.6), by looking at the pseudoperiod of $\tilde{f}_i(\boldsymbol{\lambda})$. Let $P(\boldsymbol{\lambda}) = (\gamma_1, \gamma_2, \dots, \gamma_e)$.

The operator $\tilde{f}_i$ acts on $\boldsymbol{\lambda}$ either by:

1. Adding a node to a part α whose rightmost node belongs to $P(\boldsymbol{\lambda})$. This is the setting of Lemma 5.4.17. Let $\gamma = \gamma_k = (a, \alpha, c)$ be the node of $P(\boldsymbol{\lambda})$ such that γ^+ is the good addable i-node of $\boldsymbol{\lambda}$. In this case, we have

   $$P(\psi(\boldsymbol{\lambda})) = (\gamma_1, \gamma_2, \dots, \gamma_{k-1}, \delta, \gamma_{k+1}, \dots, \gamma_e),$$

   where:

   - $\delta = (b, \alpha, c+1)$, with $b = a + N_{c+1}^{>\alpha} + N_{c+1}^{\alpha} - N_c^{>\alpha}$, if $c < l$, and
   - $\delta = (b, \alpha, 1)$, with $b = a + N_1^{>\alpha} + N_1^{\alpha} - N_l^{>\alpha}$, if $c = l$.

   Indeed, this node δ is the same as the one determined in the proof of Lemma 5.4.17 (and whose canonically associated node is Δ). The value of the row b is simply computed using the fact that:

   (a) there is no part α above the part of rightmost node γ,

   (b) all parts α above the part of rightmost node δ in $\tilde{f}_i(\boldsymbol{\lambda})$ have an element of $P(\boldsymbol{\lambda})$ as rightmost node.

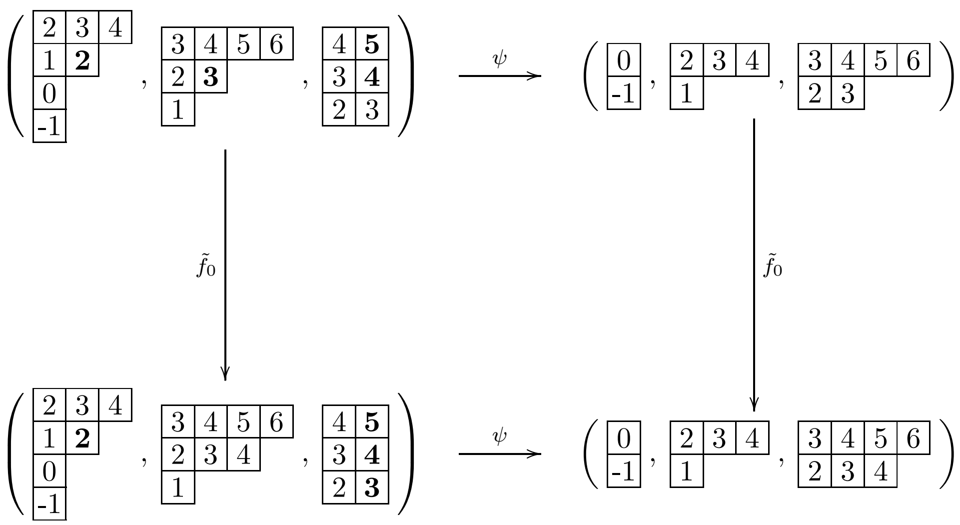
   But the part of $\tilde{f}_i(\boldsymbol{\lambda})$ whose rightmost node is γ^+ is a part of size greater than α, and is is therefore shifted to the $(c+1)$-th component (if $c < l$), or the first component (if $c = l$) when building $\psi(\tilde{f}_i(\boldsymbol{\lambda}))$. Hence, by deleting the elements of $P(\psi(\boldsymbol{\lambda}))$ and shifting the parts greater than α, we end up with the same multipartition as $\tilde{f}_i(\psi(\boldsymbol{\lambda}))$, whence the identity $\tilde{f}_i(\psi(\boldsymbol{\lambda})) = \psi(\tilde{f}_i(\boldsymbol{\lambda}))$. This is illustrated in Example 5.4.21 below.

2. Adding a node to a part $\leq \alpha$ whose rightmost node does not belong to $P(\boldsymbol{\lambda})$. This is the setting of Lemma 5.4.18. In this case , we have $P(\psi(\boldsymbol{\lambda})) = P(\boldsymbol{\lambda})$. It is then straightforward that $\tilde{f}_i(\psi(\boldsymbol{\lambda})) = \psi(\tilde{f}_i(\boldsymbol{\lambda}))$.

3. Adding a node to a part $> \alpha$ (whose rightmost node necessarily does not belong to $P(\boldsymbol{\lambda})$). This is the setting of Lemma 5.4.19. Here, we also have $P(\psi(\boldsymbol{\lambda})) = P(\boldsymbol{\lambda})$, as in the previous point.

□

Example 5.4.21. We take the same example as A.0.8, 5., namely $\boldsymbol{\lambda} = (3.2.1^2, 4.2.1, 2^3)$, $\mathbf{s} = (2, 3, 4)$, $e = 4$ and $i = 0$. Then we have the following constructions:



The bold contents represent the pseudoperiods. This illustrates the commutation between the operators ψ and $\tilde{f}_i$.

Remark 5.4.22. Note that the charged multipartition $|\psi(\boldsymbol{\lambda}), \xi(\mathbf{s})\rangle$ that we get is not cylindric anymore in general, because $\xi(\mathbf{s})$ might not be in $\mathscr{S}_e^l$. If it is, then it is clear that $\psi(\boldsymbol{\lambda})$ has one pseudoperiod less that $\boldsymbol{\lambda}$.

Remark 5.4.23. Interestingly, this isomorphism ψ can be seen as a generalisation of the cyclage isomorphism ξ. Indeed, ξ would be the version of ψ for pseudoperiods of width 0 (which can be found in any multipartition, considering that they have infinitely many parts of size 0).

By a simple use of the cyclage isomorphism, we can now easily determine a refinement Ψ of the reduction isomorphism ψ which maps a cylindric multipartition to another cylindric multipartition with one pseudoperiod less.

Let $\boldsymbol{\lambda} \in \mathscr{C}_{\mathbf{s}}$. If $\xi(\mathbf{s}) \notin \mathscr{S}_e^l$, denote by p the number of components of $\xi(\mathbf{s})$ equal to $\xi(\mathbf{s})_l$. By Remark 5.4.9 and the proof of Proposition 5.4.11, $\xi^{1+p}(\mathbf{s}) \in \mathscr{S}_e^l$.

Define $\Psi(\boldsymbol{\lambda})$ and $\Psi(\mathbf{s})$ in the following way:

- If $\xi(\mathbf{s}) \in \mathscr{S}_e^l$, then $\Psi(\boldsymbol{\lambda}) := \psi(\boldsymbol{\lambda})$ and $\Psi(\mathbf{s}) := \xi(\mathbf{s})$
- If $\xi(\mathbf{s}) \notin \mathscr{S}_e^l$, then $\Psi(\boldsymbol{\lambda}) := (\xi^p \circ \psi)(\boldsymbol{\lambda})$ and $\Psi(\mathbf{s}) := \xi^{1+p}(\mathbf{s})$.

We denote $\Psi : |\boldsymbol{\lambda}, \mathbf{s}\rangle \longmapsto |\Psi(\boldsymbol{\lambda}), \Psi(\mathbf{s})\rangle$. Then by construction, the following result holds:

Proposition 5.4.24. *For all $\boldsymbol{\lambda} \in \mathscr{C}_{\mathbf{s}}$, we have $\Psi(\boldsymbol{\lambda}) \in \mathscr{C}_{\Psi(\mathbf{s})}$. Moreover, Ψ is a $\mathcal{U}_q'(\widehat{\mathfrak{sl}_e})$-crystal isomorphism, and $\Psi(\boldsymbol{\lambda})$ has one pseudoperiod less than $\boldsymbol{\lambda}$.*

We can now determine the canonical crystal isomorphism φ for cylindric multipartitions. Recall that we have already determined $\varphi(\mathbf{s})$ in Proposition 5.4.11. It writes $\varphi(\mathbf{s}) = \xi^k(\mathbf{s})$ for some k explicitly determined.

Remark 5.4.25. The integer t defined in the proof of Proposition 5.4.11 is simply the number of pseudoperiods in $\boldsymbol{\lambda}$.

Denote t the number of pseudoperiods in $\boldsymbol{\lambda}$. Applying t times Ψ to $\boldsymbol{\lambda}$, we end up with a FLOTW multipartition, but charged by an element $\Psi^t(\mathbf{s})$ which might not be in $\mathscr{D}_e^l$. We now simply need to adjust it by some iterations of the cyclage isomorphism ξ. Since $\Psi^t(\mathbf{s}) \in \mathscr{S}_e^l$, we are ensured that $\varphi(\mathbf{s}) = (\xi^u \circ \Psi^t)(\mathbf{s})$ for some $u \in \mathbb{Z}$ easily computable.

Hence, we set
$$\varphi(\boldsymbol{\lambda}) := (\xi^u \circ \Psi^t)(\boldsymbol{\lambda}),$$
and the following theorem is straightforward.

Theorem 5.4.26. *Let $|\dot{\boldsymbol{\lambda}}, \mathbf{s}\rangle$ be a cylindric l-partition. The map*

$$\begin{array}{cccc} \varphi : & B(\dot{\boldsymbol{\lambda}}, \mathbf{s}) & \longrightarrow & B(\varphi(\dot{\boldsymbol{\lambda}}), \varphi(\mathbf{s})) \\ & \boldsymbol{\lambda} & \longmapsto & \varphi(\boldsymbol{\lambda}) \end{array}$$

is the canonical $\mathcal{U}_q'(\widehat{\mathfrak{sl}_e})$-crystal isomorphism for cylindric multipartitions.

Remark 5.4.27. Note that to determine $\varphi(\mathbf{s})$, we could also have built first $|\Psi^t(\boldsymbol{\lambda}), \Psi^t(\mathbf{s})\rangle$, and found u such that $\xi^u(\Psi^t(\mathbf{s})) \in \mathscr{D}_e^l$. Then, we would have set $\varphi(\mathbf{s}) = \xi^u(\Psi^t(\mathbf{s}))$. Clearly, this construction would give the same multicharge as the construction of $\varphi(\mathbf{s})$ in Section 5.4.2. The point of Section 5.4.2 is to show that the suitable multicharge is directly computable using only cyclages of $\mathbf{s}$.

We can therefore express the canonical crystal isomorphism Φ in full generality. Starting from any charged multipartition $|\boldsymbol{\lambda}, \mathbf{s}\rangle$, we first apply $\mathbf{RS}$, we get $\boldsymbol{\lambda}' = \mathbf{RS}(\boldsymbol{\lambda})$. Then, according to Section 5.3.3, there is an integer m such that $(\mathbf{RS} \circ \xi)^m(\boldsymbol{\lambda}')$ is cylindric. Finally, we use Ψ^t to delete all pseudoperiods and adjust everything using ξ^u so that we end up in the fundamental domain $\mathscr{D}_e^l$ (Theorem 5.4.26). It is easy to see that t does not depend on $\boldsymbol{\lambda}$ and is constant along the crystal, since it is just the number of pseudoperiods of $\boldsymbol{\lambda}$. Moreover, m does not depend on $\boldsymbol{\lambda}$ because of Proposition 5.3.12. Hence, neither does u, since it is just the "adjusting" multiplicity of ξ. This gives a generic expression for Φ, regardless of the multipartition $\boldsymbol{\lambda}$ we start with, and depending only of the connected component $B(\dot{\boldsymbol{\lambda}}, \mathbf{s})$. With the notation $\varphi = \xi^u \circ \Psi^t$, this gives the following result:

Corollary 5.4.28. *Let $\mathbf{s} \in \mathbb{Z}^l$ and $B(\overset{\bullet}{\boldsymbol{\lambda}}, \mathbf{s})$ be a connected component of the crystal graph of $\mathcal{F}_{\mathbf{s}}$. Then the canonical crystal isomorphism is*

$$\Phi = \varphi \circ (\boldsymbol{RS} \circ \xi)^m \circ \boldsymbol{RS}.$$

Remark 5.4.29. In concrete terms, since the data of $B(\overset{\bullet}{\boldsymbol{\lambda}}, \mathbf{s})$ is given by the l-charge $\mathbf{s}$ and some vertex $\boldsymbol{\lambda} \in B(\overset{\bullet}{\boldsymbol{\lambda}}, \mathbf{s})$, one can determine the expression of Φ by making these manipulations on $|\boldsymbol{\lambda}, \mathbf{s}\rangle$. If we know the highest weight vertex $\overset{\bullet}{\boldsymbol{\lambda}}$, it is natural to take $\boldsymbol{\lambda} = \overset{\bullet}{\boldsymbol{\lambda}}$ to compute Φ.

Illustrations for the crystal graph isomorphisms induced by ξ, $\mathbf{RS}$ and ψ can be found in the last section of Appendix B.

5.5 An application

Let $\mathbf{s} \in \mathbb{Z}^l$. In this last section, we deduce a non-recursive characterisation of all the vertices of any connected component of $B(\mathcal{F}_{\mathbf{s}})$.

Fix $B(\overset{\bullet}{\boldsymbol{\lambda}}, \mathbf{s})$ a connected component of $B(\mathcal{F}_{\mathbf{s}})$. This implies that we know one of the vertices of $B(\overset{\bullet}{\boldsymbol{\lambda}}, \mathbf{s})$. We assume without loss of generality (see Remark 5.4.29) that we know $\overset{\bullet}{\boldsymbol{\lambda}}$. Then the expression of the canonical crystal isomorphism

$$\begin{array}{rccc} \Phi: & B(\overset{\bullet}{\boldsymbol{\lambda}}, \mathbf{s}) & \longrightarrow & B(\mathbf{r}) \\ & |\boldsymbol{\lambda}, \mathbf{s}\rangle & \longmapsto & |\boldsymbol{\mu}, \mathbf{r}\rangle \end{array}$$

is obtained by manipulating $|\overset{\bullet}{\boldsymbol{\lambda}}, \mathbf{s}\rangle$. Corollary 5.4.28 shows the three basic crystal isomorphisms needed to construct Φ, namely $\mathbf{RS}$, ξ, and ψ (keeping in mind that $\varphi = \xi^k \circ \psi^t$ for some t and k). Amongst them, ξ is the only map which is clearly invertible. However, it is possible, keeping extra information, to make $\mathbf{RS}$ and ψ invertible.

5.5.1 Invertibility of the crystal isomorphism RS

First, it is well known (e.g. [42]) that the correspondence $\mathrm{read}(\boldsymbol{\lambda}, \mathbf{s}) \longmapsto \mathscr{P}(\mathrm{read}(\boldsymbol{\lambda}, \mathbf{s}))$ becomes a bijection if we also associate to $\mathrm{read}(\boldsymbol{\lambda}, \mathbf{s})$ its "recording symbol", i.e. the symbol with the same shape as $\mathscr{P}(\mathrm{read}(\boldsymbol{\lambda}, \mathbf{s})) =: \mathscr{P}$ in which we put the entry k in the spot where a letter appears at the k-th step. We denote by $\mathscr{Q}(\mathrm{read}(\boldsymbol{\lambda}, \mathbf{s}))$ or simply $\mathscr{Q}$ this symbol.

Example 5.5.1. Take $\mathbf{s} = (2, 0)$ and $\boldsymbol{\lambda} = (3.2, 3^2)$. Then

$$\mathfrak{B}_{\mathbf{s}}(\boldsymbol{\lambda}) = \begin{pmatrix} 0 & 4 & 5 & & \\ 0 & 1 & 2 & 5 & 7 \end{pmatrix}.$$

We get $\mathrm{read}(\boldsymbol{\lambda}, \mathbf{s}) = 54075210$. We can thus give the sequence of symbols leading to $\mathscr{P}$, and, on the right, the corresponding recording symbols, leading to $\mathscr{Q}$.

$$\begin{pmatrix} 5 \end{pmatrix} \qquad\qquad \begin{pmatrix} 1 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 5 \end{pmatrix} \qquad\qquad \begin{pmatrix} 1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 4 & 5 \end{pmatrix} \qquad\qquad \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 4 & 5 \\ 7 & & \end{pmatrix} \qquad\qquad \begin{pmatrix} 1 & 2 & 3 \\ 4 & & \end{pmatrix}$$

$$\begin{pmatrix} 0 & 4 & 5 \\ 5 & 7 & \end{pmatrix} \qquad\qquad \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & \end{pmatrix}$$

$$\begin{pmatrix} 0 & 4 & 5 \\ 2 & 5 & 7 \end{pmatrix} \qquad\qquad \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 2 & 4 & 5 \\ 1 & 5 & 7 & \end{pmatrix} \qquad\qquad \begin{pmatrix} 1 & 2 & 3 & 7 \\ 4 & 5 & 6 & \end{pmatrix}$$

$$\mathscr{P} = \begin{pmatrix} 0 & 1 & 2 & 4 & 5 \\ 0 & 5 & 7 & & \end{pmatrix} \qquad\qquad \mathscr{Q} = \begin{pmatrix} 1 & 2 & 3 & 7 & 8 \\ 4 & 5 & 6 & & \end{pmatrix}$$

Therefore, this extra data $\mathscr{Q}$ turns **RS** into a bijection.

5.5.2 Invertibility of the reduction isomorphism

Recall that the l-tuple charging $\psi(\boldsymbol{\lambda})$ is nothing but $\xi(\mathbf{s})$. Hence, the computation of $\mathbf{s}$ from $\psi(\mathbf{s}) = \xi(\mathbf{s})$ is straightforward. Moreover, starting from $\psi(\boldsymbol{\lambda})$, it is easy to recover the l-partition $\boldsymbol{\lambda}$ provided we know the width α of the pseudoperiod that has been deleted. In fact:

1. This data determines which parts will stay in the same component (namely the ones smaller than or equal to α), and which will be shifted "to the left" (namely the ones greater than α). Moreover, the property on the contents (Proposition 5.4.15), which says that all nodes must keep the same content, ensures that we can keep the boxes filled in with the same integers.

2. It remains to insert the e parts of the α-pseudoperiod at the right locations, i.e. so that the diagram obtained is in fact the Young diagram of a charged multipartition (which is possible thanks, again, to Proposition 5.4.15).

Example 5.5.2. We take, as in Example 5.4.14, $e = 4$ and

$$\psi(\boldsymbol{\lambda}) = \left(\begin{ytableau} 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 4 \\ 2 \\ 1 \end{ytableau} \; , \; \begin{ytableau} 5 & 6 & 7 & 8 & 9 & 10 \\ 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 4 \\ 2 \\ 1 \end{ytableau} \; , \; \begin{ytableau} 6 & 7 & 8 \\ 5 \\ 4 \\ 3 \end{ytableau} \right),$$

and we suppose that we know the width of $P(\boldsymbol{\lambda})$, namely $\alpha = 2$. Then the two steps above give:

1. Shifting the parts greater that α and keeping the same filling:

$$\left(\begin{ytableau} 5 & 6 & 7 & 8 & 9 & 10 \\ 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 4 \\ 2 \\ 1 \end{ytableau} , \begin{ytableau} 6 & 7 & 8 \\ 3 & 4 \\ 2 \\ 1 \end{ytableau} , \begin{ytableau} 8 & 9 & 10 & 11 & 12 & 13 \\ 5 \\ 4 \\ 3 \end{ytableau} \right).$$

   Note that at this point, this object cannot be seen as a charged l-partition (the entries in the boxes are not proper contents).

2. Inserting coherently the 4 missing parts of size 2 (represented in bold type):

$$\left(\begin{ytableau} 5 & 6 & 7 & 8 & 9 & 10 \\ 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 4 \\ 2 \\ 1 \end{ytableau} , \begin{ytableau} 6 & 7 & 8 \\ \mathbf{5} & \mathbf{6} \\ \mathbf{4} & \mathbf{5} \\ 3 & 4 \\ 2 \\ 1 \end{ytableau} , \begin{ytableau} 8 & 9 & 10 & 11 & 12 & 13 \\ \mathbf{7} & \mathbf{8} \\ \mathbf{6} & \mathbf{7} \\ 5 \\ 4 \\ 3 \end{ytableau} \right),$$

   which is indeed equal to $\boldsymbol{\lambda}$.

Hence, this extra data α turns ψ into an invertible map.

5.5.3 A non-recursive characterisation of the vertices of any connected component

Both **RS** and ψ being turned into bijections with the appropriate extra data, we can turn the canonical crystal isomorphism $\Phi : B(\overset{\bullet}{\boldsymbol{\lambda}}, \mathbf{s}) \longrightarrow B(\mathbf{r})$ into an invertible map. Concretely, this is achieved by collecting the recording data when computing Φ by manipulating $|\overset{\bullet}{\boldsymbol{\lambda}}, \mathbf{s}\rangle$. According to Corollary 5.4.28, this recording data consists of a pair $(\underline{\mathscr{Q}}, \underline{\alpha})$, where

- $\underline{\mathscr{Q}}$ is an $(m+1)$-tuple of recording symbols $(\mathscr{Q}_0, \mathscr{Q}_1, \dots, \mathscr{Q}_m)$ (corresponding to the occurences of **RS** in Φ), and
- $\underline{\alpha}$ is a t-tuple of integers $(\alpha_1, \dots, \alpha_t)$ (the widths of the different pseudoperiods),

where t and m are such that $\Phi = \xi^u \circ \Psi^t \circ (\mathbf{RS} \circ \xi)^m \circ \mathbf{RS}$.

We write $\Phi^{-1} : B(\mathbf{r}) \longrightarrow B(\overset{\bullet}{\boldsymbol{\lambda}}, \mathbf{s})$ for the inverse map.

Now, since the vertices of $B(\mathbf{r})$ are FLOTW l-partitions, they have an explicit, non-recursive characterisation. Hence, we have the following non-recursive characterisation of all the vertices of $B(\overset{\bullet}{\boldsymbol{\lambda}}, \mathbf{s})$:

Theorem 5.5.3. *The set of vertices of* $B(\overset{\bullet}{\boldsymbol{\lambda}}, \mathbf{s})$ *is equal to*

$$\left\{ \Phi^{-1}(\boldsymbol{\mu}) \; ; \; \boldsymbol{\mu} \in B(\mathbf{r}) \right\}.$$

Remark 5.5.4. Following Remark 5.4.29, if we do not know $\overset{\bullet}{\boldsymbol{\lambda}}$ but some other $\boldsymbol{\lambda} \in B(\overset{\bullet}{\boldsymbol{\lambda}}, \mathbf{s})$ instead, one can still determine Φ and Φ^{-1}. Then Theorem 5.5.3 enables the direct computation of the highest weight vertex, namely $\overset{\bullet}{\boldsymbol{\lambda}} = \Phi^{-1}(\boldsymbol{\emptyset})$.

Remark 5.5.5. This gives an analogue of the Robinson-Schensted-Knuth correspondence [108]: we have a one-to-one correspondence

$$|\boldsymbol{\lambda}, \mathbf{s}\rangle \overset{1-1}{\longleftrightarrow} (|\boldsymbol{\mu}, \mathbf{r}\rangle, (\underline{\mathscr{Q}}, \underline{\alpha}))$$

between the set of charged l-partitions on the one hand, and the set of pairs consisting of an FLOTW l-partition $|\boldsymbol{\mu}, \mathbf{r}\rangle$ and a recording data $(\underline{\mathscr{Q}}, \underline{\alpha})$ on the other hand.

Alternatively, $(\underline{\mathscr{Q}}, \underline{\alpha})$ can be replaced by $|\overset{\bullet}{\boldsymbol{\lambda}}, \mathbf{s}\rangle$, since the recording data is entirely determined by the highest weight vertex, for which a characterisation was obtained in [79].

Remark 5.5.6. All of the algorithms presented above have been implemented in Python (in particular, they are usable with the software Sage).

Perspectives

In this last part, we present some possible developments, using the results and the techniques exposed in this thesis. We indicate four different directions towards which we could orientate further research.

Identical rows in decomposition matrices

In his paper [129], Wildon has proved that

- in odd characteristic, the decomposition matrices of symmetric groups have distinct rows.
- in characteristic 2, the rows indexed by λ and μ are identical if and only if λ and μ are conjugate.

Moreover, this result holds for Hecke algebras of symmetric groups specialised at a primitive root of unity whose order is the above characteristic.

It is natural to wonder if this still holds for the groups $W_n = G(l,1,n)$ and the corresponding Ariki-Koike algebras $\mathbf{H}_{k,n}$. Actually, Wildon's proof in the case $l=1$ uses the explicit determination of the decomposition map, via the Schur functions, which is no longer valid in higher level. However, recall that one can compute the decomposition matrix D of $\mathbf{H}_{k,n} = \mathbf{H}_{k,n}^{(e,\mathbf{s})}$ using Uglov's canonical basis of the Fock space $\mathcal{F}_{\mathbf{s}}$. Denote $\Delta(q)$ the square matrix this canonical basis decomposed on the basis of l-partitions, and $D(q)$ the matrix of the canonical basis of the irreducible highest weight module $V(\mathbf{s}) = \mathcal{U}_q(\widehat{\mathfrak{sl}_e})|\boldsymbol{\emptyset}, \mathbf{s}\rangle$. By Ariki's theorem 3.3.1, we have $D(1) = D$. Hence, in addition to D, one can ask these questions for the matrices $\Delta(q)$ and $D(q)$.

Remember from the discussion at the beginning of paragraph 4.4.2 that it has been proved by Uglov [126] that the matrix $\Delta(q)$ is always unitriangular with respect to the order $\leq_{\mathcal{U}}$. Since it is a square matrix, it obviously can never have two identical rows. Hence the questions are relevant only for $D(q)$ and D. One could then use Jacon's algorithm [76] to compute these matrices and check if Wildon's criterion still holds in the case of Ariki-Koike algebras. Besides, if one manages to find two identical rows in $D(q)$, then the two corresponding rows of D will also be identical, since they are just specialisations at $q=1$ of the rows of $D(q)$. However, it is not clear (and presumably not true) that the converse will hold : if there are two identical rows in D, then it is not necessarily the case in $D(q)$.

Some examples of decomposition matrices have been computed in Appendix C. One of the first observations is that it is more likely to find two equal rows in level greater than 1 : this seems to happen whenever the multicharge $\mathbf{s}$ has symmetry properties it would be interesting to investigate.

Plactic relations in affine type A

In finite type A, the Robinson-Schensted-Knuth (RSK) correspondence has a natural interpretation in terms of crystals. It associates to any word w on $\{1, 2, \dots, e\}$ of length n a pair $(P(w), Q(w))$ of Young tableaux with common shape (of rank n) and such that $P(w)$ is semistandard, and $Q(w)$ is standard. Denote by V_e the vector representation of $\mathfrak{sl}_e$. By considering each vertex in the crystal graph of the representation $\bigoplus_{n\geq 0} V_e^{\otimes n}$ as a word on $\{1, 2, \dots, e\}$, we have

- $P(w_1) = P(w_2)$ if and only if $w_1 \sim w_2$, i.e. if and only if w_1 and w_2 appear at the same place in two isomorphic connected components (in other terms, this gives the canonical crystal isomorphism in finite type A, see Corollary 5.2.2), and
- $Q(w_1) = Q(w_2)$ if and only if w_1 and w_2 appear in the same connected component.

We refer to [101] for details.

Now, it is well-known that the equivalence relation $\sim$ is described by the so-called *Knuth* (or plactic) relations. More precisely, the quotient of the free monoid $\mathcal{A}_e^*$ on $\{1, 2, \dots, e\}$ by the relation $\sim$ is also the quotient of $\mathcal{A}_e^*$ by the relations :

$$\begin{array}{rcrcl} zxy = xyz & \text{and} & yzx = yxz & \text{if} & x < y < z \\ xyx = xxy & \text{and} & xyy = yxy & \text{if} & x < y, \end{array}$$

which we call the Knuth relations, see for instance [108].

Since we have described the equivalence relation $\sim$ in the affine case in Chapter 5, via the canonical crystal isomorphism Φ, it is natural to wonder if one can obtain similar relations in this setting. The problem here is that because of the affine feature, the cyclage operator ξ (which is ubiquitous in the construction of Φ) is not compatible with the monoid structure of $\mathcal{A}_e^*$, given by concatenation of words. However, it might be interesting to look for analogues of these elementary plactic relations in this case.

Littelmann paths and the Pitman transform in affine type A

We have already mentioned in the introduction that there is a tight connection between the Kashiwara crystals, the Littelmann paths, and the Pitman transform, see for instance [11] or [103]. However, most results are established in the case of finite Coxeter groups. Our study of crystals in affine type A could lead to similar results in this more general case. The questions one could ask are:

1. How to deduce from the canonical crystal isomorphism of Chapter 5 an explicit description for the Pitman transform in affine type A?

2. How is Uglov's realisation of the crystal related to the Littelmann path model?

According to the work of Biane, Bougerol and O'Connell [11], [12], determining the Pitman transform comes down to finding the highest weight vertex in the corresponding crystal, which is precisely what is done in Remark 5.5.4.

As for the second question, it is not clear how both structure intertwine, since with Uglov's realisation, the Fock space is, contrary to Kleshchev's realisation, not a tensor product of level 1 Fock spaces, which is needed to interpret the concatenation of Littelmann paths. However, both realisations of $\mathcal{F}_{\mathbf{s}}$ being isomorphic, it should also be possible to find a connection.

Harish-Chandra branching graphs for unitary groups and Hiss' conjectures

Let $G = GU_n(p)$ be the unitary group on the finite field with p elements. The representation theory of G is complicated, but there exists a class of simple modules which is quite well understood: the unipotent modules. They are parametrised by the set of partitions of n:

$$\{Y_\lambda \ ; \ \lambda \vdash n\},$$

see [112].

By the works of Lusztig [109] and Fong and Srinivasan [41], we have a nice combinatorial characterisation of the Harish-Chandra series in the ordinary case, namely

$$Y_\lambda \text{ and } Y_\mu \text{ are in the same HC series} \quad \Leftrightarrow \quad \lambda_{(2)} = \mu_{(2)}, \tag{5.7}$$

where $\lambda_{(2)}$ is the 2-core of λ.

If d is a prime number dividing the order of G but not p, then we are interested in the d-modular representation theory of G. We have in particular a set of d-modular unipotents modules, labelled again by the partitions of n

$$\{X_\lambda \ ; \ \lambda \vdash n\},$$

according to [43]. It is then natural to ask for similar results as (5.7) in the d-modular case.

To each integer $m \leq n$ is associated a *Levi subgroup* L_m, which is isomorphic to $GU_m(p) \times GL_1(p^2)^{\frac{n-m}{2}}$. We can then see the unipotent module Y_λ as an L_m-module, where $\lambda \vdash m$, by identifying it with $Y_\lambda \otimes 1$. Now, the so-called *Harish-Chandra (HC) induction* is a procedure that associates a G-module to any L_m-module, see [72]. Write $R_{L_m}^G$ for this induction functor. It has an adjoint functor which we denote by ${}^*R_{L_m}^G$ (*HC-restriction*). A cuspidal module is a module M such that ${}^*R_{L_m}^G(M) = 0$. Moreover, according to [72], we have a partition of the set of unipotent modules into *HC series*, each of which consisting of the head composition factors of an HC-induced cuspidal module.

Denote e the order of $-p$ modulo d, i.e. $e = \min\{k \in \mathbb{Z}_{>0} \text{ such that } d|(-p)^k - 1\}$. When e is even, the description of the HC series is known thanks to the works of [50] and [65]. In the case where e is odd, Hiss has conjectured in 2009 that

Conjecture 1.

1. *X_λ and X_μ are in the same HC series* $\Rightarrow \lambda_{(2)} = \mu_{(2)}$.

2. *X_λ cuspidal* $\Rightarrow$ *the e-core of λ is a 2-core.*

This implies that the partition into HC series in the modular case is a refinement of the partitions into HC series in the ordinary case.

Now, some observations show similarities between Harish-Chandra theory and Kashiwara's crystal theory. We define the *Harish-Chandra branching graph* to be the graph with

- vertices: all partitions of n whose rank have the same parity (therefore, this defines two different sets of vertices),

- arrows: $\lambda \longrightarrow \mu$ if and only if $|\mu| = |\lambda|+2$ and X_μ is a submodule of $R_{L_n}^{GU_{n+2}(p)}(X_\lambda)$, where $n = |\lambda|$.

For $t \in \mathbb{Z}_{>0}$, let Δ_t be the triangular partition $(t, t-1, \dots, 1)$. Then the integer $|\Delta_t|$ is called a triangular number. For a triangular number $|\Delta_t|$ consider the subgraph of the Harish-Chandra branching graph with vertices the partitions whose 2-core is Δ_t. Now, replace the labels of these vertices by their 2-quotients. Denote $H(t)$ the corresponding graph. Then, Hiss [70] has made the following observation.

Conjecture 2. *The graph $H(t)$ coincides with the crystal graph $B(\mathcal{F}_\mathbf{s})$, with $\mathbf{s} = (t + (1-e)/2, 0)$, up to rank d.*

Remark 5.5.7. The Fock space $\mathcal{F}_\mathbf{s}$ here is considered as a $\mathcal{U}_q(\widehat{\mathfrak{sl}_e})$-module, where e is the odd integer defined above.

Note that here, we are interested only in the oriented graph structure of $B(\mathcal{F}_\mathbf{s})$, without considering the labelling of the arrows.

In particular, the cuspidal modules would be labeled by highest weight vertices in the crystal graph, for which we have an explicit characterisation, see [79]. Moreover, for a highest weight vertex $\boldsymbol{\lambda}$ in $B(\boldsymbol{\lambda}, \mathbf{s})$, the Kashiwara crystal would yield a labelling of the unipotent modules in the HC series of X_λ in the groups $GU_{|\boldsymbol{\lambda}|+2r}(p)$ for $r \geq 0$, where $\boldsymbol{\lambda}$ is the 2-quotient of λ.

Some computations have showed that this conjecture is true for $n \leq 10$ and $e = 3, 5$, as well as $n = 12$ and $e = 3$. Moreover, it is known that this holds for the connected component containing Δ_t under some extra assumptions, and that some consequences of this conjecture are true. This indicates some interesting developments. Indeed, if conjecture 2 was proved, then one could rephrase conjecture 1 in terms of crystals, which might be a more fruitful approach.

Appendix

Appendix A

Proof of Lemmas 5.4.17 , 5.4.18 and 5.4.19

Proof of Lemma 5.4.17. The proof first splits in three cases (which, in turn split in subcases), even though ultimately, the argument is the same. In each case (and subcase), we determine a certain node δ of $\boldsymbol{\lambda}$ which is not in $P(\boldsymbol{\lambda})$. Then, we show that the node of $\psi(\boldsymbol{\lambda})$ canonically associated to δ is the node Δ we expect.

1. Assume first that γ is the first element of $P(\boldsymbol{\lambda})$. Denote by $\gamma_e = (a_e, \alpha, c_e)$ the last node of $P(\boldsymbol{\lambda})$. Then $\mathrm{cont}(\gamma_e) = \mathrm{cont}(\gamma^+) - e$, and since γ^+ is an i-node, then so is γ_e.

   (a) Suppose that γ_e is removable (cf Example A.0.8, 1.). Since γ^+ is the good addable i-node, the letter R produced by γ_e in the i-word must not simplify with the A produced by γ^+. This means that there exists an addable node $\tilde{\gamma}^+$ in $\lambda^{\tilde{c}}$ with either

   - $\tilde{c} > c$ and $\mathrm{cont}(\tilde{\gamma}^+) = \mathrm{cont}(\gamma^+)$, or
   - $\tilde{c} < c_e$ and $\mathrm{cont}(\tilde{\gamma}^+) = \mathrm{cont}(\gamma^+) - e$ (one cannot have $\tilde{c} = c_e$ since otherwise γ_e would not be removable).

   In the first case, this means that there is an integer β in the $\tilde{c}$-th row of $\mathfrak{B}_{\mathbf{s}}(\boldsymbol{\lambda})$ which is also in the c-th row and the same column. By the semistandardness of $\mathfrak{B}_{\mathbf{s}}(\boldsymbol{\lambda})$ (because $|\boldsymbol{\lambda}, \mathbf{s}\rangle$ is cylindric), we are ensured that β is also present in the row d and the same column for all $d \in [\![c, \tilde{c}]\!]$. This is equivalent to saying that there is a part of size α in each component λ^d with $d \in [\![c, \tilde{c}]\!]$ whose rightmost node $\hat{\gamma}$ verifies $\mathrm{cont}(\hat{\gamma}) = \mathrm{cont}(\gamma)$. We denote by δ the one located in the component λ^{c+1}. In particular, δ^+ is an i-node. Moreover, δ^+ is an addable node of $\boldsymbol{\lambda}$. Indeed, if it is not, then there is a part of size α just above the part whose rightmost node is $\tilde{\gamma}$. Denote by $\overline{\gamma}$ its rightmost node. Then $\mathrm{cont}(\overline{\gamma}) = \mathrm{cont}(\delta) + 1 = \mathrm{cont}(\gamma) + 1$. This implies that $\overline{\gamma}$ must be in $P(\boldsymbol{\lambda})$, which contradicts the fact that γ is the first element of $P(\boldsymbol{\lambda})$.

   In the second case, there is an integer β in the $\tilde{c}$-th row of $\mathfrak{B}_{\mathbf{s}}(\boldsymbol{\lambda})$ which is also in the c_e-th row and the same column, say column k. Again, since the symbol is semistandard, the elements $\mathfrak{b}_{d,k}$ appearing in row d, with $d \in [\![1, \tilde{c}]\!]$, and in column k verify $\mathfrak{b}_d \geq \mathfrak{b}_{d'} \geq \beta$ for $1 \leq d' < d < c_e$. But the cylindricity property also implies that $\mathfrak{b}_{l,k+e} \geq \mathfrak{b}_{1,k} + e$. Actually, the $(k+e)$-th column of the symbol is also the column that contains the integer corresponding to the first node of $P(\boldsymbol{\lambda})$, since pseudoperiods have length e. Moreover, this element is equal to $\beta + e$. In other terms, $\mathfrak{b}_{c,k+e} = \beta + e$. By semistandardness again, one must have $\beta + e \geq \mathfrak{b}_{d,k+e} \geq \mathfrak{b}_{d',k+e}$ for all $c \leq d < d' \leq l$. To sum up, we have

$$\begin{aligned} \beta + e \quad = \quad & \mathfrak{b}_{c,k+e} \geq \mathfrak{b}_{c+1,k+e} \geq \cdots \geq \mathfrak{b}_{l,k+e} \geq \mathfrak{b}_{1,k} + e \geq \dots \\ & \qquad \geq \mathfrak{b}_{c_e-1,k} + e \geq \mathfrak{b}_{c_e,k} + e = \beta + e, \end{aligned}$$

thus all inequalities are in fact equalities. As in the first case, this means in particular that there is a part of size α in λ^{c+1} (if $c < l$) or in λ^1 (if $c = l$) whose rightmost node δ verifies $\mathrm{cont}(\delta) = \mathrm{cont}(\gamma) - e$. In particular, δ^+ is an i-node. Moreover, δ^+ is an addable node of $\boldsymbol{\lambda}$. Indeed, if it is not, then there is a part of size α just above the part whose rightmost node is $\tilde{\gamma}$. Denote by $\overline{\gamma}$ its rightmost node. Then $\mathrm{cont}(\overline{\gamma}) = \mathrm{cont}(\delta) + 1 = \mathrm{cont}(\gamma) - e + 1 = \mathrm{cont}(\gamma_e)$. This implies that $\overline{\gamma}$ must be the last element of $P(\boldsymbol{\lambda})$ instead of γ_e, which is a contradiction.

(b) Suppose that γ_e is not removable (cf Example A.0.8, 2.). This means that there exists a part of size α below the part whose rightmost node is γ_e. Note that the rightmost node $\tilde{\gamma}$ of this part has content $\mathrm{cont}(\tilde{\gamma}) = \mathrm{cont}(\gamma_e) - 1$. By the same cylindricity argument used in 1.(a), this part of size α spreads in all components of $\boldsymbol{\lambda}$. This means that there exists a part α in λ^{c+1} (if $c < l$) or in λ^1 (if $c = l$) with rightmost node δ verifying $\mathrm{cont}(\delta) = \mathrm{cont}(\gamma)$ (if $c < l$) and $\mathrm{cont}(\delta) = \mathrm{cont}(\gamma) - e$ (if $c = l$). In particular, δ^+ is an i-node. Moreover, if δ^+ is addable, unless, of course, it is in the component λ^{c_e}. This can be seen using the exact same argument as in 1.(a).

2. Assume now that γ is the last element of $P(\boldsymbol{\lambda})$. First of all, note that if $l > 1$, one can never have $c = l$. Indeed, in this case γ^+ would not be an addable node.

   (a) Suppose that $s_{c+1} > s_c$ (cf Example A.0.8, 3.). Then, using Proposition 5.4.6, we can claim that there exists a part of size α in the component λ^{c+1}. Denote $\tilde{\gamma}$ its rightmost node. By Lemma 5.4.3, $\mathrm{cont}(\tilde{\gamma}) = \mathrm{cont}(\gamma) + 1 = \mathrm{cont}(\gamma^+)$, and $\tilde{\gamma}$ is an i-node. Now, if $\tilde{\gamma}$ is removable, then $\tilde{\gamma}$ and γ^+ yield an occurence of RA which contradicts the fact that γ^+ is the good i-node of type A. Hence, there is necessarily a part α below the part whose rightmost node is $\tilde{\gamma}$. Denote by δ its rightmost node, so that δ^+ is an i-node.

   (b) Suppose now that $s_{c+1} = s_c$ (cf Example A.0.8, 4.). Consider the previous node in $P(\boldsymbol{\lambda})$, denote it by γ_{e-1}. By Lemma 5.4.3 again, $\mathrm{cont}(\gamma_{e-1}) = \mathrm{cont}(\gamma)+1 = \mathrm{cont}(\gamma^+)$, so that γ_{e-1} is an i-node. Besides, since γ^+ is addable, γ_{e-1} is removable unless there is a part α below the part whose rightmost node is γ_{e-1}. But if γ_{e-1} is removable, then there exists an addable i-node $\tilde{\gamma}$ in $\lambda^{\tilde{c}}$ with $\tilde{c} \in [\![c+1, \dots, c_1 - 1]\!]$, where c_1 is the component of $\boldsymbol{\lambda}$ which contains the first node of $P(\boldsymbol{\lambda})$ (otherwise γ^+ and γ_{e-1} yield and occurence RA and γ^+ cannot be the good addable i-node). Then, by the cylindricity argument again, there is a part α in each component λ^d with $d \in [\![c+1,\dots,\tilde{c}]\!]$, whose rightmost node has content $\mathrm{cont}(\gamma)$. In particular, this is true for $d=c+1$. We denote δ the one located in the component λ^{c+1}.

Now if γ_{e-1} is not removable, i.e. if there exists a part α below the part whose rightmost node is γ_{e-1}, then again the cylindricity implies that this parts spreads to all components λ^d with $d \in [\![c+1,\dots,c_{e-1}-1]\!]$, where c_{e-1} is the component of $\boldsymbol{\lambda}$ which contains γ_{e-1}, and have the same contents. Again, we denote δ the rightmost node of the part α of λ^{c+1} which has content $\mathrm{cont}(\delta)=\mathrm{cont}(\gamma)$.

3. Assume finally that γ is neither the first nor the last node of $P(\boldsymbol{\lambda})$. Again, c cannot be equal to l because then it would be the first node of $P(\boldsymbol{\lambda})$.

   (a) Suppose that $s_{c+1} > s_c$ (cf Example A.0.8, 5.). We can use the same arguments as in 2.(a), and define δ in the exact same way.

   (b) Suppose that $s_c = s_{c+1}$ (cf Example A.0.8, 6 .). Then since γ^+ is not the first node of $P(\boldsymbol{\lambda})$, there exists a part α in a component $\lambda^{\tilde{c}}$ with $\tilde{c} > c$ whose rightmost node $\tilde{\gamma}$ has content $\mathrm{cont}(\tilde{\gamma}) = \mathrm{cont}(\gamma)+1$. If $\tilde{c}=c+1$, then we can use the previous case 3.(a). If $\tilde{c} > c+1$, then:

   - if $\tilde{\gamma}$ is removable, then there exists a part α in a component $\bar{c}$ with $c < \bar{c} < \tilde{c}$ whose rightmost node has content $\mathrm{cont}(\gamma)$, otherwise $\tilde{\gamma}$ produces, together with γ^+, an occurence RA, whence the usual contradiction. By cylindricity, such a part α also exists in the $(c+1)$-th component of $\boldsymbol{\lambda}$. We denote δ its rightmost node.
   - if $\tilde{\gamma}$ is not removable, then there exists a part α below the part whose rightmost node is $\tilde{\gamma}$, with rightmost content equal to $\mathrm{cont}(\gamma)$. Again, by cylindricity, it also exists in the $(c+1)$-th component of $\boldsymbol{\lambda}$, and we denote δ its rightmost node.

It is obvious, but important to notice, that δ is not a node of $P(\boldsymbol{\lambda})$. Hence, there is a node Δ of $\psi(\boldsymbol{\lambda})$ which is canonically associated to δ. By Proposition 5.4.15, $\mathrm{cont}_{\psi(\boldsymbol{\lambda})}(\Delta) = \mathrm{cont}_{\boldsymbol{\lambda}}(\delta)$ and in fact $\Delta = (a,\alpha,c+1)$ if $c<l$ and $\Delta=(a,\alpha,1)$ if $c=l$. In particular Δ^+ is an i-node. Moreover, it is addable in $\psi(\boldsymbol{\lambda})$ since δ^+ is either addable in $\boldsymbol{\lambda}$, or the rightmost node of a part above which sit parts that are deleted after applying ψ.

In fact, Δ^+ is the good addable i-node of $\psi(\boldsymbol{\lambda})$. Indeed, consider the i-word for $\psi(\boldsymbol{\lambda})$. Denote it by w_i^ψ, and denote w_i the i-word for $\boldsymbol{\lambda}$. By construction of $\psi(\boldsymbol{\lambda})$, the subword of w_i corresponding to the rightmost nodes of the parts that are either greater than α (respectively smaller than or equal to α but not in $P(\boldsymbol{\lambda})$) is also a subword of w_i^ψ. The only differences that are likely to appear are the following:

- The letters R and A that correspond to nodes in $P(\boldsymbol{\lambda})$ vanish. Note that we have assumed that there is always such a letter A in w_i (since γ is in $P(\boldsymbol{\lambda})$).

- The parts of size α that are below a part whose rightmost node is in $P(\boldsymbol{\lambda})$ give a new letter A in $\psi(\boldsymbol{\lambda})$.

Now by construction of δ:

1. If δ^+ gives a letter A in w_i, then it is adjacent to the letter A encoding γ^+, to its left. Hence, since the A encoding γ^+ is no longer in w_i^ψ, the letter A corresponding to Δ^+ in w_i^ψ plays the same role as the one corresponding to γ^+ in $\boldsymbol{\lambda}$: it is the rightmost A in the reduced i-word of $\psi(\boldsymbol{\lambda})$. In other terms, Δ^+ is the good addable i-node of $\psi(\boldsymbol{\lambda})$.

2. If δ^+ does not give a letter A in w_i, that is if there is an element of $P(\boldsymbol{\lambda})$ just above δ, then it is clear that, again, the A encoding Δ^+ in w_i^ψ plays the same role as the A encoding γ^+ in w_i, and that Δ^+ is the good addable i-node of $\psi(\boldsymbol{\lambda})$.

□

Example A.0.8.

1. $\boldsymbol{\lambda} = (4.2, 2^2, 5.2)$, $\mathbf{s} = (2, 3, 4)$, $e = 3$ and $i = 2$.

2. $\boldsymbol{\lambda} = (6^2.2.1^3, 4.2^3.1^2, 6.2^2.1^4)$, $\mathbf{s} = (5, 6, 8)$, $e = 4$ and $i = 1$.

3. $\boldsymbol{\lambda} = (2.1, 1^3, 1)$, $\mathbf{s} = (3, 4, 5)$, $e = 4$ and $i = 3$.

4. $\boldsymbol{\lambda} = (1, 1, 1^3)$, $\mathbf{s} = (4, 4, 7)$, $e = 4$ and $i = 1$.

5. $\boldsymbol{\lambda} = (3.2.1^2, 4.2.1, 2^3)$, $\mathbf{s} = (2, 3, 4)$, $e = 4$ and $i = 0$.

6. $\boldsymbol{\lambda} = (2^2, 3.2, 2)$, $\mathbf{s} = (3, 4, 4)$, $e = 3$ and $i = 0$.

Proof of Lemma 5.4.18. First, note that Γ is nothing but the node of $\psi(\boldsymbol{\lambda})$ canonically associated to γ in the definition of ψ. Besides, by definition of $\psi(\boldsymbol{\lambda})$, together with Proposition 5.4.15, we are ensured that Γ^+ is in fact an addable i-node of $\psi(\boldsymbol{\lambda})$. Moreover, w_i^ψ is likely to contain new letters A, namely the one corresponding to nodes η^+ with $\eta = (b, \alpha, d) \notin P(\boldsymbol{\lambda})$ lying just below a node of $P(\boldsymbol{\lambda})$. Denote by H the node of $\psi(\boldsymbol{\lambda})$ corresponding to η. Note that $\mathrm{cont}(\eta) \neq \max_{\tilde{\gamma} \in P(\boldsymbol{\lambda})}(\mathrm{cont}(\tilde{\gamma}))$. Indeed, since there is a node of $P(\boldsymbol{\lambda})$ just above η, it is not the first node of $P(\boldsymbol{\lambda})$ and hence it does not have maximal content. Since $\eta \notin P(\boldsymbol{\lambda})$, there exists a node $\tilde{\gamma} = (\tilde{a}, \alpha, \tilde{c}) \in P(\boldsymbol{\lambda})$ such that $\mathrm{cont}(\tilde{\gamma}) = \mathrm{cont}(\eta)$, $\tilde{c} < d$, and $\tilde{\gamma}^+$ is addable. Hence H plays the same role in $\psi(\boldsymbol{\lambda})$ as $\tilde{\gamma}$ in $\boldsymbol{\lambda}$. In particular, since the good addable i-node of $\boldsymbol{\lambda}$ is in a part of size smaller than α, there is necessarily a letter R, encoding a node ρ, that simplifies with the letter A encoding $\tilde{\gamma}^+$. Now:

- If ρ is the rightmost node of a part of size different than α, then it is obviously not in $P(\boldsymbol{\lambda})$.

- If ρ is the rightmost node of a part of size α and $\mathrm{cont}(\rho) = \mathrm{cont}(\eta^+)$, then it is not in $P(\boldsymbol{\lambda})$ either. Indeed, there is a node of $P(\boldsymbol{\lambda})$ just above η whose content is $\mathrm{cont}(\eta^+)$, which is therefore not of type R, and hence different from ρ, and since all nodes of $P(\boldsymbol{\lambda})$ have different contents, ρ is not in $P(\boldsymbol{\lambda})$.

- If ρ is the rightmost node of a part of size α and $\mathrm{cont}(\rho) < \mathrm{cont}(\eta^+)$, then it is not in $P(\boldsymbol{\lambda})$ either, because the contents of the nodes of $P(\boldsymbol{\lambda})$ are consecutive (cf. Lemma 5.4.3), and because ρ does not have maximal content.

Therefore, the letter R encoding ρ is also present in w_i^ψ, and simplifies with the A encoding H. This implies that Γ^+ is the good addable i-node of $\psi(\boldsymbol{\lambda})$.

□

Proof of Lemma 5.4.19. As in Lemma 5.4.18, Γ is the node of $\psi(\boldsymbol{\lambda})$ canonically associated to γ. Consider the letter A encoding γ^+. It is the rightmost letter A in w_i and does not simplify. By Proposition 5.4.15, Γ^+ is also encoded by a letter A which the righmost letter A in w_i^ψ. It remains to show that it does not simplify with any letter R either. In fact, as noticed in the previous proofs, the deletion of the pseudoperiod, in the construction of $\psi(\boldsymbol{\lambda})$, cannot yield any new letter R. However, some letters A encoding nodes of $P(\boldsymbol{\lambda})$ can vanish. Denote by η^+ such a node.

Suppose first that η is not the first node of $P(\boldsymbol{\lambda})$. Since η^+ is addable, there cannot be another element of $P(\boldsymbol{\lambda})$ above η. Then denote by η_1 the node of $P(\boldsymbol{\lambda})$ which has content $\mathrm{cont}(\eta_1) = \mathrm{cont}(\eta) + 1$ (i.e. the previous node of $P(\boldsymbol{\lambda})$). Note that if η is in λ^d, then η_1 is in λ^{d+1}. Suppose now that η is the first node of $P(\boldsymbol{\lambda})$. Then, similarly, consider the last node of $P(\boldsymbol{\lambda})$ and denote it by $P(\boldsymbol{\lambda})$. In each case, either:

- there is a part α just below the part whose rightmost node is η_1, in which case the node just below η_1 is not in $P(\boldsymbol{\lambda})$ and yields an addable node in $\psi(\boldsymbol{\lambda})$ which plays exactly the same role in $\psi(\boldsymbol{\lambda})$ as η in $\boldsymbol{\lambda}$;

- or there is no node just below η_1, in which case η_1 is encoded by a letter R which simplifies with the letter A encoding η^+.

As a consequence, we are ensured that the letter A encoding Γ^+ does not simplify in w_i^ψ, and hence Γ^+ is the good addable i-node of $\psi(\boldsymbol{\lambda})$.

□

Appendix B

Crystal graphs

Crystal graphs of irreducible highest weight modules

Example B.0.9. Take $l = 3$, $\mathbf{s} = (-1, -1, 0)$ and $e = 2$. The beginning of the crystal graph $B(\boldsymbol{\emptyset}, \mathbf{s})$ of the irreducible highest weight module $V(\mathbf{s})$ looks as follows.

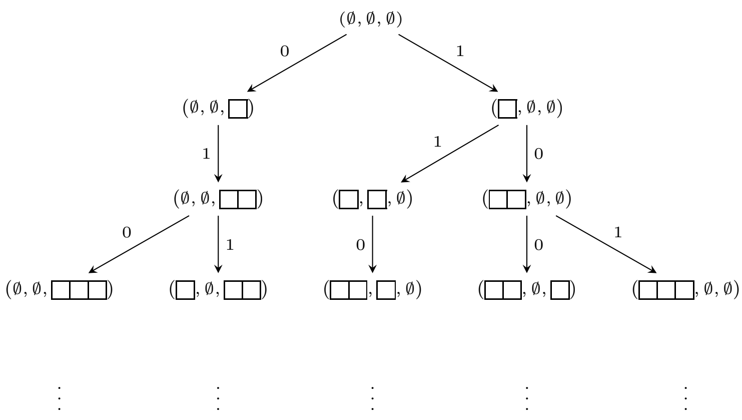


Figure B.1: The crystal graph of $V(\mathbf{s})$ for $l = 3$, $\mathbf{s} = (-1, -1, 0)$ and $e = 2$.

Its vertices are the Uglov l-partitions $\Phi_{\mathbf{s}}$. Note that in this case, $\mathbf{s} \in \mathscr{S}_e^l$, hence $\Phi_{\mathbf{s}} = \Psi_{\mathbf{s}}$ (i.e. the Uglov multipartitions are actually the FLOTW multipartitions). By Proposition 3.2.6, $B(\boldsymbol{\emptyset}, \mathbf{s})$ is isomorphic to $B(\boldsymbol{\emptyset}, \mathbf{r})$ whenever $\mathbf{r} \in \mathcal{C}(\mathbf{s})$. For instance, take $\mathbf{r} = (-1, 0, -1)$ and $\mathbf{t} = (5, -3, 0)$. We can compute the associated crystal graphs:

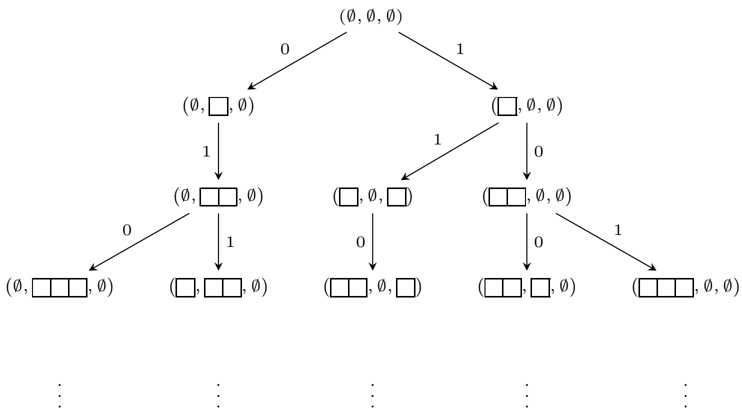


Figure B.2: The crystal graph of $V(\mathbf{r})$ for $l = 3$, $\mathbf{r} = (-1, 0, -1)$ and $e = 2$.

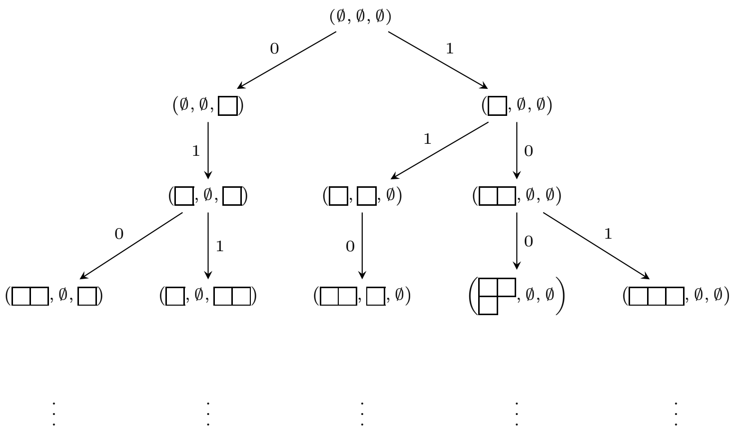


Figure B.3: The crystal graph of $V(\mathbf{t})$ for $l = 3$, $\mathbf{t} = (5, -3, 0)$ and $e = 2$.

We see that $B(\boldsymbol{\emptyset}, \mathbf{r})$ and $B(\boldsymbol{\emptyset}, \mathbf{t})$ are both isomorphic to $B(\boldsymbol{\emptyset}, \mathbf{s})$ (but not equal).

Example B.0.10. This is to illustrate Example 4.3.3.

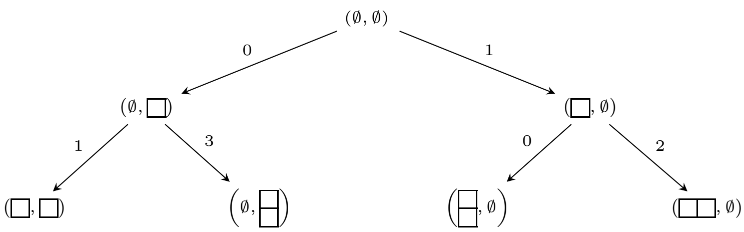


Figure B.4: Uglov l-partitions for $l = 2$, $\mathbf{s} = (1, 0)$ and $e = 4$ up to rank 2.

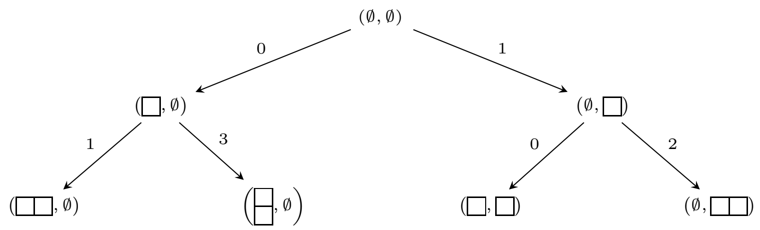


Figure B.5: Uglov l-partitions for $l = 2$, $\mathbf{r} = (0, 1)$ and $e = 4$ up to rank 2.

We see that $\Phi_{\mathbf{s}}(2) \neq \Phi_{\mathbf{r}}(2)$.

Twisted Uglov multipartitions

Level 2

By the discussion following Proposition 4.3.5, we must have

$$\Phi_{(s_1, s_2+e)}(n) = \{(\lambda^2, \lambda^1) \,;\, (\lambda^1, \lambda^2) \in \Phi_{(s_2,s_1)}(n)\},$$

that is, the set of twisted Uglov bipartitions is itself a set of Uglov bipartitions. This is illustrated in the examples below.

Example B.0.11.

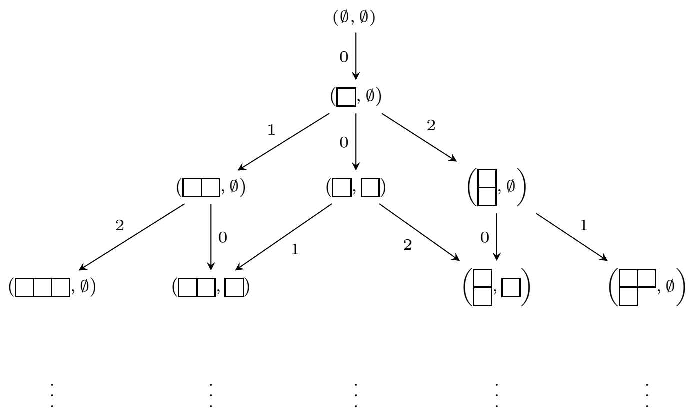


Figure B.6: Uglov bipartitions for $\mathbf{s} = (0, 0)$ and $e = 3$.

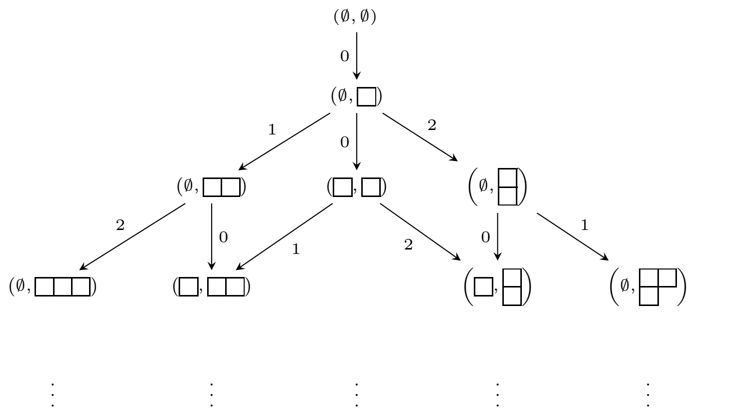


Figure B.7: Uglov bipartitions for $\tilde{\mathbf{s}} = (0, 3)$ and $e = 3$.

We remark that the set of Uglov bipartitions associated to $\tilde{\mathbf{s}}$ is simply the set of twisted Uglov bipartition associated to $\mathbf{s}$ (where $\tilde{\mathbf{s}} = (s_2, s_1 + e)$).

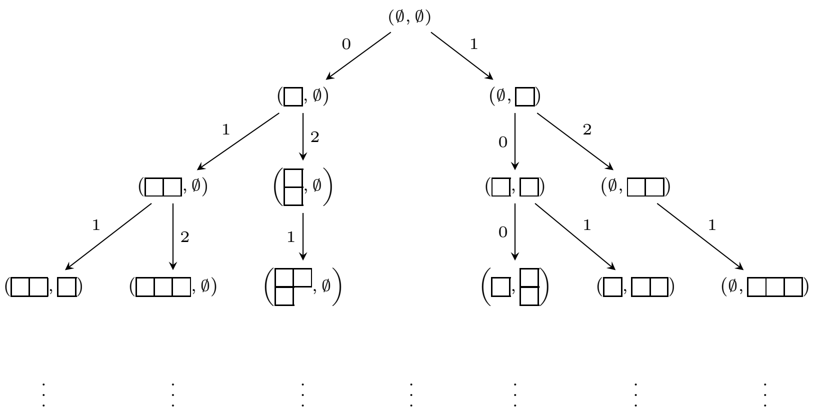


Figure B.8: Uglov bipartitions for $\mathbf{r} = (0, 1)$ and $e = 3$.

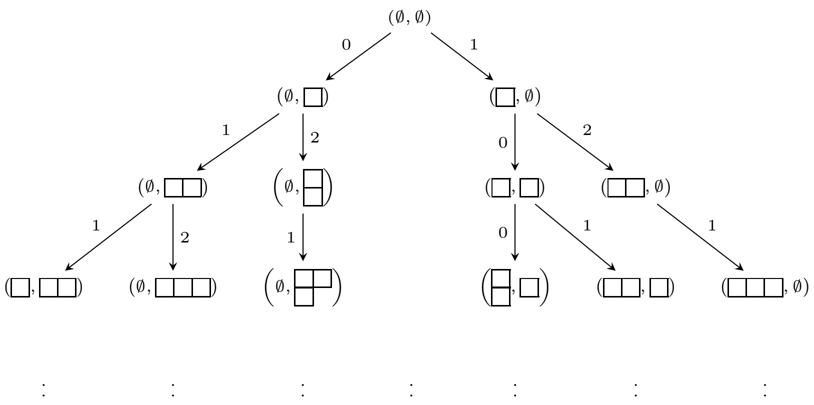


Figure B.9: Uglov bipartitions for $\tilde{\mathbf{r}} = (1, 3)$ and $e = 3$.

Again, we remark that the set of Uglov bipartitions associated to $\tilde{\mathbf{r}}$ is simply the set of twisted Uglov bipartition associated to $\mathbf{r}$ (where $\tilde{\mathbf{r}} = (r_2, r_1 + e)$).

Level 3

In higher level, twisting multipartitions does not necessarily correspond to a crystal isomorphism, as mentionned on page 90. In other terms, the σ-twisted Uglov multipartitions are not, in general, Uglov multipartitions, but we know exactly when this is the case. In level 3, Example 4.3.7 says that this happens when $\sigma \in \{\mathrm{Id}, (123), (132)\}$.

Example B.0.12. Let us illustrate this phenomenon for $\sigma = (123)$.

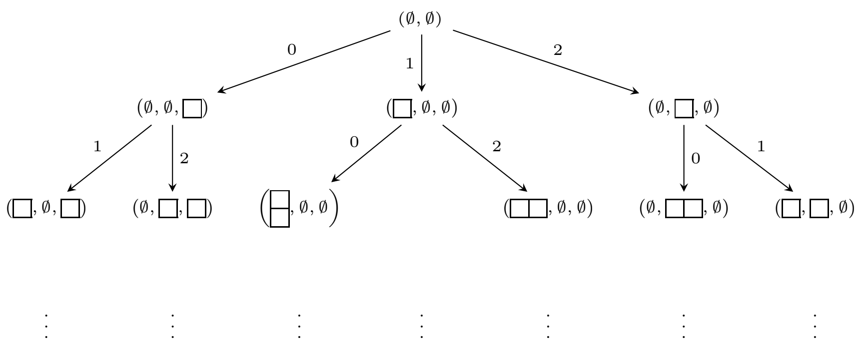


Figure B.10: Uglov 3-partitions for $\tilde{\mathbf{s}} = (2, 0, 4)$ and $e = 3$.

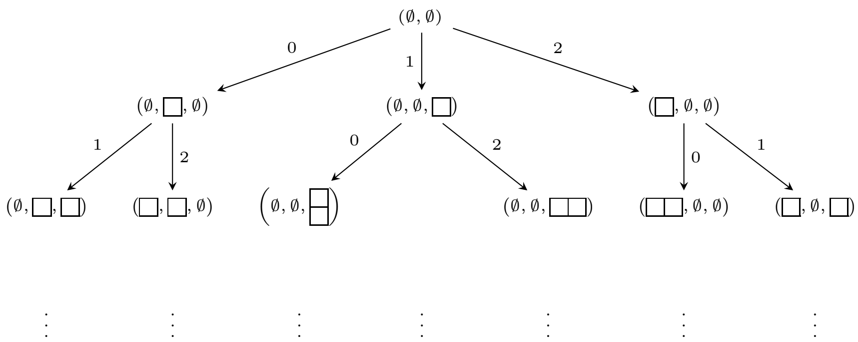


Figure B.11: Uglov 3-partitions for $\mathbf{s} = (1, 2, 0)$ and $e = 3$.

We see that the Uglov 3-partitions associated to $\tilde{\mathbf{s}}$ are exactly the σ-twisted Uglov 3-partitions associated to $\mathbf{s}$.

Crystal isomorphisms between arbitrary connected components

In this section, we illustrate the fact that the maps ξ, $\mathbf{RS}$, and ψ are crystal isomorphisms. We take $l = 3$, $\mathbf{s} = (2, 1, 5)$ and $\boldsymbol{\lambda} = \overset{\bullet}{\boldsymbol{\lambda}} = (\emptyset, \emptyset, 1^3)$, and we apply successively ξ,

RS, and ψ.

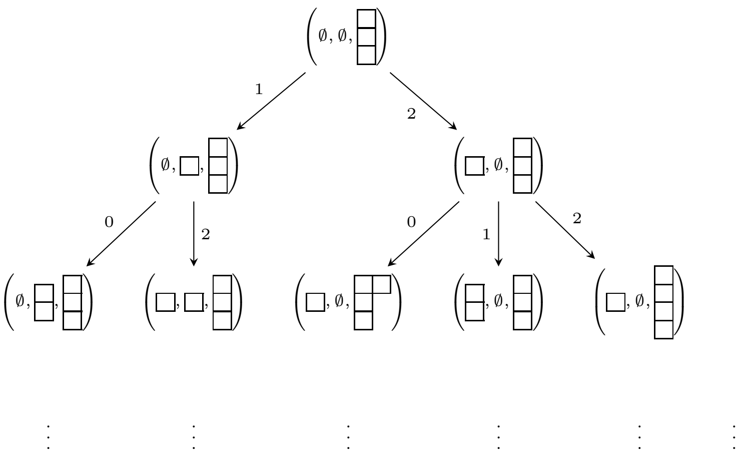


Figure B.12: The crystal graph $B(\boldsymbol{\lambda}, \mathbf{s})$.

The cyclage isomorphism

Denote $\boldsymbol{\mu} = \xi(\boldsymbol{\lambda}) = (1^3, \emptyset, \emptyset)$ and $\mathbf{r} = \xi(\mathbf{s}) = (2, 2, 1)$.

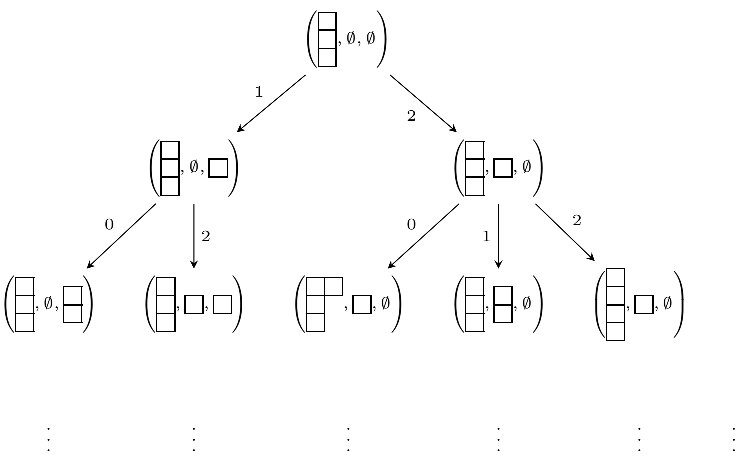


Figure B.13: The crystal graph $B(\boldsymbol{\mu}, \mathbf{r})$.

The isomorphism RS

Denote $\boldsymbol{\nu} = \mathbf{RS}(\boldsymbol{\mu})$ and $\mathbf{t} = \mathbf{RS}(\mathbf{r})$. One can check that $\boldsymbol{\nu} = (1^2, 1, \emptyset)$ and $\mathbf{t} = (1, 2, 2)$.

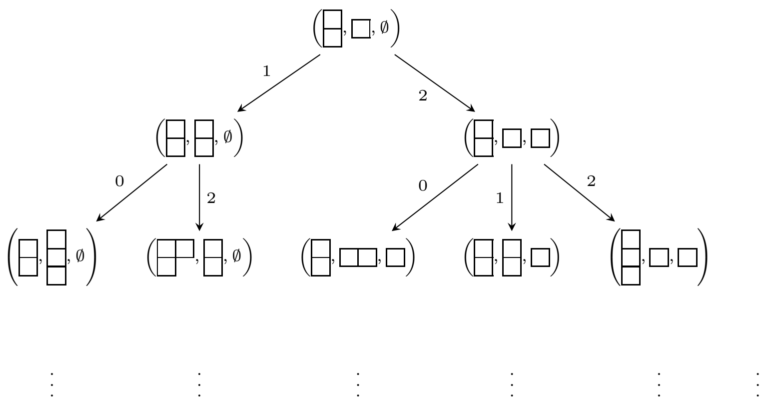


Figure B.14: The crystal graph $B(\boldsymbol{\nu}, \mathbf{t})$.

The reduction isomorphism

One can check that the resulting multipartition $|\boldsymbol{\nu}, \mathbf{t}\rangle$ is actually cylindric. Therefore, one can apply the reduction isomorphism ψ to it. Denote $\boldsymbol{\pi} = \psi(\boldsymbol{\nu})$ and $\mathbf{v} = \psi(\mathbf{t})$. We have $\boldsymbol{\pi} = (\emptyset, \emptyset, \emptyset)$ and $\mathbf{v} = (-1, 1, 2)$.

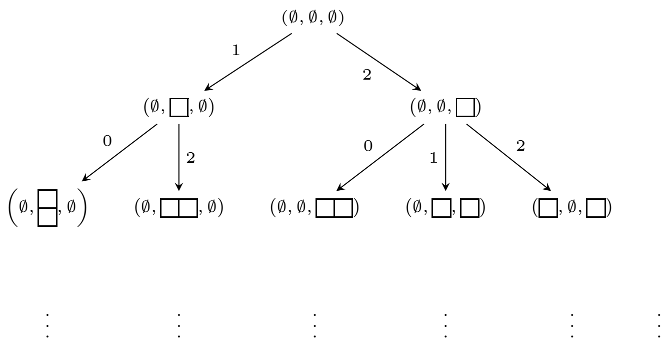


Figure B.15: The crystal graph $B(\boldsymbol{\pi}, \mathbf{v})$.

We end up with a set of Uglov 3-partitions. Note that these are not FLOTW 3-partitions because the 3-charge is not in $\mathscr{S}_3^3$. We can get the FLOTW 3-partitions by applying one more time the cyclage ξ.

Appendix C

Decomposition matrices of Ariki-Koike algebras

In this appendix, we have computed, using Jacon's algorithm [76], some decomposition matrices of Ariki-Koike algebras. The dots stand for zeros. For each example, we specify the level l, the multicharge $\mathbf{s}$, and the integers e and n. We use multicharges in $\mathscr{S}_e^l$, so that the columns are parametrised by the FLOTW multipartitions. We order the rows and columns so that the k-th row and the k-th column are labelled by the same FLOTW multipartition (for $k = 1, 2, \ldots, |\Psi_{\mathbf{s}}(n)|$), and that the matrix has a unitriangular shape (see Theorem 4.2.2).

Level 2

- $e = 2$, $\mathbf{s} = (0, 0)$, $n = 1, 2, 3, 4$.

$$\begin{array}{c||c} 1 & (1,\emptyset) \\ 1 & (\emptyset,1) \end{array}$$

$$\begin{array}{cc||c} 1 & . & (2,\emptyset) \\ . & 1 & (1,1) \\ 1 & . & (1^2,\emptyset) \\ 1 & . & (\emptyset,2) \\ 1 & . & (\emptyset,1^2) \end{array}$$

$$\begin{array}{ccc||c} 1 & . & . & (3,\emptyset) \\ . & 1 & . & (2.1,\emptyset) \\ 1 & . & 1 & (2,1) \\ 1 & . & . & (\emptyset,3) \\ 1 & . & 1 & (1,2) \\ 1 & . & 1 & (1^2,1) \\ 1 & . & . & (1^3,\emptyset) \\ 1 & . & 1 & (1,1^2) \\ . & 1 & . & (\emptyset,2.1) \\ 1 & . & . & (\emptyset,1^3) \end{array}$$

$$\begin{array}{cccc||c} 1 & . & . & . & (4,\emptyset) \\ . & 1 & . & . & (3,1) \\ 1 & . & 1 & . & (3.1,\emptyset) \\ 1 & . & . & 1 & (2,2) \\ 1 & . & . & . & (\emptyset,4) \\ . & . & 1 & . & (2^2,\emptyset) \\ . & 1 & . & . & (1,3) \\ 1 & . & 1 & 1 & (2.1,1) \\ 1 & . & 1 & . & (2.1^2,\emptyset) \\ 1 & . & . & 1 & (2,1^2) \\ 1 & . & 1 & . & (\emptyset,3.1) \\ 1 & . & . & 1 & (1^2,2) \\ 1 & . & 1 & 1 & (1,2.1) \\ . & 1 & . & . & (1^3,1) \\ 1 & . & . & 1 & (1^2,1^2) \\ . & . & 1 & . & (\emptyset,2^2) \\ 1 & . & . & . & (1^4,\emptyset) \\ 1 & . & 1 & . & (\emptyset,2.1^2) \\ . & 1 & . & . & (1,1^3) \\ 1 & . & . & . & (\emptyset,1^4) \end{array}$$

- $e = 2$, $\mathbf{s} = (0, 1)$, $n = 1, 2, 3, 4$.

1	.	$(1,\emptyset)$
.	1	$(\emptyset,1)$

1	.	$(2,\emptyset)$
.	1	$(\emptyset,2)$
1	1	$(1,1)$
1	.	$(1^2,\emptyset)$
.	1	$(\emptyset,1^2)$

1	.	.	.	$(3,\emptyset)$
.	1	.	.	$(\emptyset,3)$
1	.	1	.	$(1,2)$
.	1	.	1	$(2,1)$
.	.	.	1	$(2.1,\emptyset)$
.	.	1	.	$(\emptyset,2.1)$
.	1	.	1	$(1^2,2)$
1	.	1	.	$(1,1^2)$
1	.	.	.	$(1^3,\emptyset)$
.	1	.	.	$(\emptyset,1^3)$

1	.	.	.	.	.	$(4,\emptyset)$
.	1	.	.	.	.	$(\emptyset,4)$
1	1	1	.	.	.	$(3,1)$
1	1	.	1	.	.	$(1,3)$
.	.	.	.	1	.	$(1,2.1)$
.	.	.	.	.	1	$(2.1,1)$
.	1	.	1	.	.	$(\emptyset,3.1)$
1	1	1	1	.	.	$(2,2)$
1	.	1	.	.	.	$(3.1,\emptyset)$
1	1	1	1	.	.	$(2,1^2)$
.	.	.	1	.	.	$(\emptyset,2^2)$
1	1	1	1	.	.	$(1^2,2)$
.	.	1	.	.	.	$(2^2,\emptyset)$
1	.	1	.	.	.	$(2.1^2,\emptyset)$
1	1	1	1	.	.	$(1^2,1^2)$
.	1	.	1	.	.	$(\emptyset,2.1^2)$
1	1	1	.	.	.	$(1^3,1)$
1	1	.	1	.	.	$(1,1^3)$
1	.	.	.	.	.	$(1^4,\emptyset)$
.	1	.	.	.	.	$(\emptyset,1^4)$

- $e=3$, $\mathbf{s}=(0,0)$, $n=1,2,3,4$.

1	$(1,\emptyset)$
1	$(\emptyset,1)$

1	.	.	$(2,\emptyset)$
.	1	.	$(1^2,\emptyset)$
.	.	1	$(1,1)$
1	.	.	$(\emptyset,2)$
.	1	.	$(\emptyset,1^2)$

1	.	.	.	$(3,\emptyset)$
1	1	.	.	$(2.1,\emptyset)$
.	.	1	.	$(2,1)$
.	.	.	1	$(1^2,1)$
1	.	.	.	$(\emptyset,3)$
.	.	1	.	$(1,2)$
.	1	.	.	$(1^3,\emptyset)$
.	.	.	1	$(1,1^2)$
1	1	.	.	$(\emptyset,2.1)$
.	1	.	.	$(\emptyset,1^3)$

1	.	.	.	.	.	.	.	$(4,\emptyset)$
.	1	.	.	.	.	.	.	$(3.1,\emptyset)$
1	.	1	.	.	.	.	.	$(3,1)$
1	.	.	1	.	.	.	.	$(2^2,\emptyset)$
.	.	.	.	1	.	.	.	$(2,2)$
1	.	1	1	.	1	.	.	$(2.1,1)$
.	.	.	.	.	.	1	.	$(2.1^2,\emptyset)$
.	.	.	.	.	.	.	1	$(1^2,1^2)$
1	.	.	.	.	.	.	.	$(\emptyset,4)$
1	.	1	.	.	.	.	.	$(1,3)$
.	.	1	.	.	1	.	.	$(1^2,2)$
.	.	1	.	.	1	.	.	$(2,1^2)$
.	1	.	.	.	.	.	.	$(\emptyset,3.1)$
1	.	1	1	.	1	.	.	$(1,2.1)$
.	.	.	1	.	1	.	.	$(1^3,1)$
.	.	.	1	.	.	.	.	$(1^4,\emptyset)$
1	.	.	1	.	.	.	.	$(\emptyset,2^2)$
.	.	.	1	.	1	.	.	$(1,1^3)$
.	.	.	.	.	.	1	.	$(\emptyset,2.1^2)$
.	.	.	1	.	.	.	.	$(\emptyset,1^4)$

- $e = 3$, $\mathbf{s} = (0, 1)$, $n = 1, 2, 3, 4$.

$$\begin{array}{cc||c}
1 & . & (1,\emptyset) \\
. & 1 & (\emptyset,1)
\end{array}
\qquad
\begin{array}{cccc||c}
1 & . & . & . & (2,\emptyset) \\
. & 1 & . & . & (\emptyset,2) \\
1 & . & 1 & . & (1,1) \\
. & . & . & 1 & (1^2,\emptyset) \\
. & . & 1 & . & (\emptyset,1^2)
\end{array}
\qquad
\begin{array}{cccccc||c}
1 & . & . & . & . & . & (3,\emptyset) \\
. & 1 & . & . & . & . & (\emptyset,3) \\
. & . & 1 & . & . & . & (2,1) \\
1 & . & . & 1 & . & . & (2.1,\emptyset) \\
1 & 1 & . & . & 1 & . & (1,2) \\
. & . & . & . & . & 1 & (1,1^2) \\
1 & . & . & 1 & 1 & . & (1^2,1) \\
. & 1 & . & . & 1 & . & (\emptyset,2.1) \\
. & . & . & 1 & . & . & (1^3,\emptyset) \\
. & . & . & . & 1 & . & (\emptyset,1^3)
\end{array}$$

$$\begin{array}{ccccccccccc||c}
1 & . & . & . & . & . & . & . & . & . & . & (4,\emptyset) \\
. & 1 & . & . & . & . & . & . & . & . & . & (\emptyset,4) \\
. & . & 1 & . & . & . & . & . & . & . & . & (3,1) \\
. & . & . & 1 & . & . & . & . & . & . & . & (3.1,\emptyset) \\
1 & . & . & . & 1 & . & . & . & . & . & . & (1,3) \\
. & 1 & 1 & . & . & 1 & . & . & . & . & . & (2,2) \\
1 & . & . & . & . & . & 1 & . & . & . & . & (2^2,\emptyset) \\
. & . & 1 & . & . & 1 & . & 1 & . & . & . & (2.1,1) \\
. & . & . & . & . & . & . & . & 1 & . & . & (2,1^2) \\
. & . & . & 1 & . & . & . & . & . & 1 & . & (1^2,2) \\
1 & . & . & . & 1 & . & . & . & . & . & 1 & (1,2.1) \\
. & . & . & . & 1 & . & . & . & . & . & . & (\emptyset,3.1) \\
. & 1 & . & . & . & 1 & . & . & . & . & . & (\emptyset,2^2) \\
. & . & . & . & . & . & . & 1 & . & . & . & (2.1^2,\emptyset) \\
1 & . & . & . & . & . & 1 & . & . & . & 1 & (1^2,1^2) \\
. & . & . & . & . & 1 & . & 1 & . & . & . & (1^3,1) \\
. & . & . & . & . & . & . & . & . & 1 & . & (\emptyset,2.1^2) \\
. & . & . & . & . & . & . & . & . & . & 1 & (1,1^3) \\
. & . & . & . & . & . & 1 & . & . & . & . & (1^4,\emptyset) \\
. & . & . & . & . & 1 & . & . & . & . & . & (\emptyset,1^4)
\end{array}$$

- $e = 4$, $\mathbf{s} = (0, 0)$, $n = 1, 2, 3, 4$.

$$\begin{array}{c||c}
1 & (1,\emptyset) \\
1 & (\emptyset,1)
\end{array}
\qquad
\begin{array}{ccc||c}
1 & . & . & (2,\emptyset) \\
. & 1 & . & (1^2,\emptyset) \\
. & . & 1 & (1,1) \\
1 & . & . & (\emptyset,2) \\
. & 1 & . & (\emptyset,1^2)
\end{array}
\qquad
\begin{array}{ccccc||c}
1 & . & . & . & . & (3,\emptyset) \\
. & 1 & . & . & . & (2.1,\emptyset) \\
. & . & 1 & . & . & (2,1) \\
. & . & . & 1 & . & (1^3,\emptyset) \\
. & . & . & . & 1 & (1^2,1) \\
. & . & 1 & . & . & (1,2) \\
1 & . & . & . & . & (\emptyset,3) \\
. & . & . & . & 1 & (1,1^2) \\
. & 1 & . & . & . & (\emptyset,2.1) \\
. & . & . & 1 & . & (\emptyset,1^3)
\end{array}$$

$$\begin{array}{ccccccccc||c}
1 & . & . & . & . & . & . & . & . & (4,\emptyset) \\
1 & 1 & . & . & . & . & . & . & . & (3.1,\emptyset) \\
. & . & 1 & . & . & . & . & . & . & (3,1) \\
. & . & . & 1 & . & . & . & . & . & (2^2,\emptyset) \\
. & 1 & . & . & 1 & . & . & . & . & (2.1^2,\emptyset) \\
. & . & . & . & . & 1 & . & . & . & (2,2) \\
. & . & . & 1 & . & . & 1 & . & . & (2.1,1) \\
. & . & . & . & . & . & . & 1 & . & (1^3,1) \\
. & . & . & . & . & . & . & . & 1 & (1^2,1^2) \\
1 & . & . & . & . & . & . & . & . & (\emptyset,4) \\
. & . & 1 & . & . & . & . & . & . & (1,3) \\
. & . & . & . & . & . & 1 & . & . & (1^2,2) \\
. & . & . & . & . & . & 1 & . & . & (2,1^2) \\
. & . & . & . & 1 & . & . & . & . & (1^4,\emptyset) \\
. & . & . & 1 & . & . & 1 & . & . & (1,2.1) \\
1 & 1 & . & . & . & . & . & . & . & (\emptyset,3.1) \\
. & . & . & 1 & . & . & . & . & . & (\emptyset,2^2) \\
. & . & . & . & . & . & . & 1 & . & (1,1^3) \\
. & 1 & . & . & 1 & . & . & . & . & (\emptyset,2.1^2) \\
. & . & . & . & 1 & . & . & . & . & (\emptyset,1^4)
\end{array}$$

- $e = 4,\ s = (0,1),\ n = 1,2,3,4.$

$$\begin{array}{cc||c}
1 & . & (1,\emptyset) \\
. & 1 & (\emptyset,1)
\end{array}$$

$$\begin{array}{cccc||c}
1 & . & . & . & (2,\emptyset) \\
. & 1 & . & . & (1^2,\emptyset) \\
1 & . & 1 & . & (1,1) \\
. & . & . & 1 & (\emptyset,2) \\
. & . & 1 & . & (\emptyset,1^2)
\end{array}$$

$$\begin{array}{cccccccc||c}
1 & . & . & . & . & . & . & . & (3,\emptyset) \\
. & 1 & . & . & . & . & . & . & (2.1,\emptyset) \\
. & . & 1 & . & . & . & . & . & (2,1) \\
. & . & . & 1 & . & . & . & . & (\emptyset,3) \\
1 & . & . & . & 1 & . & . & . & (1,2) \\
. & 1 & . & . & . & 1 & . & . & (1^2,1) \\
. & . & . & . & . & . & 1 & . & (1^3,\emptyset) \\
. & . & . & . & . & . & . & 1 & (1,1^2) \\
. & . & . & . & 1 & . & . & . & (\emptyset,2.1) \\
. & . & . & . & . & 1 & . & . & (\emptyset,1^3)
\end{array}$$

$$\begin{array}{ccccccccccccc||c}
1 & . & . & . & . & . & . & . & . & . & . & . & . & (4,\emptyset) \\
. & 1 & . & . & . & . & . & . & . & . & . & . & . & (3,1) \\
. & . & 1 & . & . & . & . & . & . & . & . & . & . & (\emptyset,4) \\
1 & . & . & 1 & . & . & . & . & . & . & . & . & . & (3.1,\emptyset) \\
. & 1 & . & . & 1 & . & . & . & . & . & . & . & . & (2,2) \\
. & . & . & . & . & 1 & . & . & . & . & . & . & . & (2^2,\emptyset) \\
1 & . & 1 & . & . & . & 1 & . & . & . & . & . & . & (1,3) \\
. & . & . & . & . & . & . & 1 & . & . & . & . & . & (2.1,1) \\
. & . & . & 1 & . & . & . & . & 1 & . & . & . & . & (2.1^2,\emptyset) \\
. & . & . & . & . & . & . & . & . & 1 & . & . & . & (2,1^2) \\
1 & . & . & 1 & . & . & 1 & . & . & . & 1 & . & . & (1^2,2) \\
. & . & . & . & . & . & . & . & . & . & . & 1 & . & (1,2.1) \\
. & . & . & . & . & 1 & . & . & . & . & . & . & 1 & (1^2,1^2) \\
. & . & 1 & . & . & . & 1 & . & . & . & . & . & . & (\emptyset,3.1) \\
. & . & . & 1 & . & . & . & . & 1 & . & 1 & . & . & (1^3,1) \\
. & . & . & . & 1 & . & . & . & . & . & . & . & . & (\emptyset,2^2) \\
. & . & . & . & . & . & . & . & 1 & . & . & . & . & (1^4,\emptyset) \\
. & . & . & . & . & . & 1 & . & . & . & 1 & . & . & (\emptyset,2.1^2) \\
. & . & . & . & . & . & . & . & . & . & . & . & 1 & (1,1^3) \\
. & . & . & . & . & . & . & . & . & . & 1 & . & . & (\emptyset,1^4)
\end{array}$$

- $e = 4,\ s = (0,2),\ n = 1,2,3,4.$

$$\begin{array}{cc||c}
1 & . & (1,\emptyset) \\
. & 1 & (\emptyset,1)
\end{array}
\qquad
\begin{array}{ccccc||c}
1 & . & . & . & . & (2,\emptyset) \\
. & 1 & . & . & . & (\emptyset,2) \\
. & . & 1 & . & . & (1,1) \\
. & . & . & 1 & . & (1^2,\emptyset) \\
. & . & . & . & 1 & (\emptyset,1^2)
\end{array}
\qquad
\begin{array}{cccccccc||c}
1 & . & . & . & . & . & . & . & (3,\emptyset) \\
. & 1 & . & . & . & . & . & . & (\emptyset,3) \\
. & 1 & 1 & . & . & . & . & . & (1,2) \\
1 & . & . & 1 & . & . & . & . & (2,1) \\
. & . & . & . & 1 & . & . & . & (2.1,\emptyset) \\
. & . & . & . & . & 1 & . & . & (\emptyset,2.1) \\
. & . & 1 & . & . & . & 1 & . & (1^2,1) \\
. & . & . & 1 & . & . & . & 1 & (1,1^2) \\
. & . & . & . & . & . & 1 & . & (1^3,\emptyset) \\
. & . & . & . & . & . & . & 1 & (\emptyset,1^3)
\end{array}$$

$$\begin{array}{ccccccccccccccc||c}
1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & (4,\emptyset) \\
. & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & (\emptyset,4) \\
. & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & (3,1) \\
. & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & (1,3) \\
. & 1 & . & . & 1 & . & . & . & . & . & . & . & . & . & . & (\emptyset,3.1) \\
1 & 1 & . & . & . & 1 & . & . & . & . & . & . & . & . & . & (2,2) \\
1 & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & (3.1,\emptyset) \\
. & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & (2,1^2) \\
. & 1 & . & . & 1 & 1 & . & . & 1 & . & . & . & . & . & . & (1,2.1) \\
. & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & (\emptyset,2^2) \\
. & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & (1^2,2) \\
1 & . & . & . & . & 1 & 1 & . & . & . & . & 1 & . & . & . & (2.1,1) \\
. & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & (2^2,\emptyset) \\
. & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & (1^3,1) \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & (1,1^3) \\
. & . & . & . & . & . & 1 & . & . & . & . & 1 & . & . & . & (2.1^2,\emptyset) \\
. & . & . & . & . & 1 & . & . & 1 & . & . & 1 & . & . & . & (1^2,1^2) \\
. & . & . & . & 1 & . & . & . & 1 & . & . & . & . & . & . & (\emptyset,2.1^2) \\
. & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & (1^4,\emptyset) \\
. & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & (\emptyset,1^4)
\end{array}$$

Level 3

- $e=2,\ \mathbf{s}=(0,0,0),\ n=1,2,3.$

$$\begin{array}{c||c}
1 & (1,\emptyset,\emptyset) \\
1 & (\emptyset,1,\emptyset) \\
1 & (\emptyset,\emptyset,1)
\end{array}
\qquad
\begin{array}{cc||c}
1 & . & (2,\emptyset,\emptyset) \\
. & 1 & (1,1,\emptyset) \\
1 & . & (\emptyset,2,\emptyset) \\
1 & . & (1^2,\emptyset,\emptyset) \\
. & 1 & (1,\emptyset,1) \\
1 & . & (\emptyset,\emptyset,2) \\
. & 1 & (\emptyset,1,1) \\
1 & . & (\emptyset,1^2,\emptyset) \\
1 & . & (\emptyset;\emptyset,1^2)
\end{array}
\qquad
\begin{array}{cccc||c}
1 & . & . & . & (3,\emptyset,\emptyset) \\
1 & 1 & . & . & (2,1,\emptyset) \\
. & . & 1 & . & (2.1,\emptyset,\emptyset) \\
. & . & . & 1 & (1,1,1) \\
1 & . & . & . & (\emptyset,3,\emptyset) \\
1 & 1 & . & . & (1,2,\emptyset) \\
1 & 1 & . & . & (2,\emptyset,1) \\
1 & . & . & . & (\emptyset,\emptyset,3) \\
1 & 1 & . & . & (1,\emptyset,2) \\
1 & 1 & . & . & (1^2,1,\emptyset) \\
1 & 1 & . & . & (\emptyset,2,1) \\
. & . & 1 & . & (\emptyset,2.1,\emptyset) \\
1 & 1 & . & . & (1^2,\emptyset,1) \\
1 & 1 & . & . & (1,1^2,\emptyset) \\
1 & 1 & . & . & (\emptyset,1,2) \\
1 & . & . & . & (1^3,\emptyset,\emptyset) \\
1 & 1 & . & . & (\emptyset,1^2,\emptyset) \\
1 & 1 & . & . & (1,\emptyset,1^2) \\
. & . & 1 & . & (\emptyset,\emptyset,2.1) \\
1 & 1 & . & . & (\emptyset,1,1^2) \\
1 & . & . & . & (\emptyset,1^3,\emptyset) \\
1 & . & . & . & (\emptyset,\emptyset,1^3)
\end{array}$$

- $e=2,\ \mathbf{s}=(0,0,1),\ n=1,2,3.$

$$\begin{array}{cc||c}
1 & . & (1,\emptyset,\emptyset) \\
. & 1 & (\emptyset,\emptyset,1) \\
1 & . & (\emptyset,1,\emptyset)
\end{array}
\qquad
\begin{array}{ccc||c}
1 & . & . & (2,\emptyset,\emptyset) \\
. & 1 & . & (\emptyset,\emptyset,2) \\
. & . & 1 & (1,1,\emptyset) \\
1 & 1 & . & (1,\emptyset,1) \\
1 & . & . & (\emptyset,2,\emptyset) \\
1 & . & . & (1^2,\emptyset,\emptyset) \\
1 & 1 & . & (\emptyset,1,1) \\
. & 1 & . & (\emptyset,\emptyset,1^2) \\
1 & . & . & (\emptyset,1^2,\emptyset)
\end{array}
\qquad
\begin{array}{ccccc||c}
1 & . & . & . & . & (3,\emptyset,\emptyset) \\
. & 1 & . & . & . & (\emptyset,\emptyset,3) \\
. & 1 & 1 & . & . & (2,\emptyset,1) \\
1 & . & . & 1 & . & (2,1,\emptyset) \\
1 & . & . & . & 1 & (1,\emptyset,2) \\
1 & . & . & . & . & (\emptyset,3,\emptyset) \\
. & . & 1 & . & . & (2.1,\emptyset,\emptyset) \\
1 & . & . & 1 & . & (1,2,\emptyset) \\
1 & . & . & . & 1 & (\emptyset,1,2) \\
. & 1 & 1 & . & . & (\emptyset,2,1) \\
2 & . & . & 1 & 1 & (1,1,1) \\
. & 1 & 1 & . & . & (1^2,\emptyset,1) \\
. & . & . & . & 1 & (\emptyset,\emptyset,2.1) \\
1 & . & . & 1 & . & (1^2,1,\emptyset) \\
1 & . & . & . & 1 & (1,\emptyset,1^2) \\
. & . & 1 & . & . & (\emptyset,2.1,\emptyset) \\
1 & . & . & 1 & . & (1,1^2,\emptyset) \\
1 & . & . & . & 1 & (\emptyset,1,1^2) \\
. & 1 & 1 & . & . & (\emptyset,1^2,1) \\
1 & . & . & . & . & (1^3,\emptyset,\emptyset) \\
. & 1 & . & . & . & (\emptyset,\emptyset,1^3) \\
1 & . & . & . & . & (\emptyset,1^3,\emptyset)
\end{array}$$

- $e=2$, $\mathbf{s}=(0,1,2)$, $n=1,2,3$.

$$\begin{array}{cc||c}
1 & . & (\emptyset,\emptyset,1) \\
. & 1 & (\emptyset,1,\emptyset) \\
1 & . & (1,\emptyset,\emptyset)
\end{array}
\qquad
\begin{array}{ccc||c}
1 & . & . & (\emptyset,\emptyset,2) \\
. & 1 & . & (\emptyset,2,\emptyset) \\
. & . & 1 & (1,\emptyset,1) \\
1 & 1 & . & (\emptyset,1,1) \\
1 & . & . & (2,\emptyset,\emptyset) \\
1 & 1 & . & (1,1,\emptyset) \\
1 & . & . & (\emptyset,\emptyset,1^2) \\
. & 1 & . & (\emptyset,1^2,\emptyset) \\
1 & . & . & (1^2,\emptyset,\emptyset)
\end{array}
\qquad
\begin{array}{ccccc||c}
1 & . & . & . & . & (\emptyset,\emptyset,3) \\
. & 1 & . & . & . & (\emptyset,3,\emptyset) \\
. & 1 & 1 & . & . & (\emptyset,1,2) \\
1 & . & . & 1 & . & (1,\emptyset,2) \\
1 & . & . & . & 1 & (\emptyset,2,1) \\
1 & . & . & . & . & (3,\emptyset,\emptyset) \\
. & . & 1 & . & . & (\emptyset,\emptyset,2.1) \\
1 & . & . & 1 & . & (2,\emptyset,1) \\
1 & . & . & . & 1 & (1,2,\emptyset) \\
. & 1 & 1 & . & . & (2,1,\emptyset) \\
2 & . & . & 1 & 1 & (1,1,1) \\
. & . & . & . & 1 & (\emptyset,2.1,\emptyset) \\
. & 1 & 1 & . & . & (\emptyset,1,1^2) \\
1 & . & . & 1 & . & (1,\emptyset,1^2) \\
1 & . & . & . & 1 & (\emptyset,1^2,2) \\
. & . & 1 & . & . & (2.1,\emptyset,\emptyset) \\
1 & . & . & 1 & . & (1^2,\emptyset,1) \\
1 & . & . & . & 1 & (1,1^2,\emptyset) \\
. & 1 & 1 & . & . & (1^2,1,\emptyset) \\
1 & . & . & . & . & (\emptyset,\emptyset,1^3) \\
. & 1 & . & . & . & (\emptyset,1^3,\emptyset) \\
1 & . & . & . & . & (1^3,\emptyset,\emptyset)
\end{array}$$

- $e=3$, $\mathbf{s}=(0,0,0)$, $n=1,2,3$.

$$
\begin{array}{c||l}
1 & (1,\emptyset,\emptyset) \\
1 & (\emptyset,1,\emptyset) \\
1 & (\emptyset,\emptyset,1)
\end{array}
\qquad
\begin{array}{ccc||l}
1 & . & . & (2,\emptyset,\emptyset) \\
. & 1 & . & (1^2,\emptyset,\emptyset) \\
. & . & 1 & (1,1,\emptyset) \\
1 & . & . & (\emptyset,2,\emptyset) \\
. & . & 1 & (1,\emptyset,1) \\
. & 1 & . & (\emptyset,1^2,\emptyset) \\
. & . & 1 & (\emptyset,1,1) \\
1 & . & . & (\emptyset,\emptyset,2) \\
. & 1 & . & (\emptyset,\emptyset,1^2)
\end{array}
\qquad
\begin{array}{ccccc||l}
1 & . & . & . & . & (3,\emptyset,\emptyset) \\
. & 1 & . & . & . & (2,1,\emptyset) \\
1 & . & 1 & . & . & (2.1,\emptyset,\emptyset) \\
. & . & . & 1 & . & (1^2,1,\emptyset) \\
. & . & . & . & 1 & (1,1,1) \\
1 & . & . & . & . & (\emptyset,3,\emptyset) \\
. & 1 & . & . & . & (1,2,\emptyset) \\
. & 1 & . & . & . & (2,\emptyset,1) \\
. & . & . & 1 & . & (1^2,\emptyset,1) \\
. & . & 1 & . & . & (1^3,\emptyset,\emptyset) \\
. & 1 & . & . & . & (\emptyset,2,1) \\
1 & . & 1 & . & . & (\emptyset,2.1,\emptyset) \\
. & . & . & 1 & . & (1,1^2,\emptyset) \\
1 & . & . & . & . & (\emptyset,\emptyset,3) \\
. & 1 & . & . & . & (1,\emptyset,2) \\
. & 1 & . & . & . & (\emptyset,1,2) \\
. & . & . & 1 & . & (\emptyset,1^2,1) \\
. & . & . & 1 & . & (1,\emptyset,1^2) \\
. & . & 1 & . & . & (\emptyset,1^3,\emptyset) \\
1 & . & 1 & . & . & (\emptyset,\emptyset,2.1) \\
. & . & . & 1 & . & (1,\emptyset,1^2) \\
. & . & 1 & . & . & (\emptyset,\emptyset,1^3)
\end{array}
$$

- $e=3,\ \mathbf{s}=(0,0,1),\ n=1,2,3.$

$$
\begin{array}{cc||l}
1 & . & (1,\emptyset,\emptyset) \\
. & 1 & (\emptyset,\emptyset,1) \\
1 & . & (\emptyset,1,\emptyset)
\end{array}
\qquad
\begin{array}{ccccc||l}
1 & . & . & . & . & (2,\emptyset,\emptyset) \\
. & 1 & . & . & . & (1^2,\emptyset,\emptyset) \\
. & . & 1 & . & . & (1,1,\emptyset) \\
1 & . & . & 1 & . & (1,\emptyset,1) \\
. & . & . & . & 1 & (\emptyset,\emptyset,2) \\
1 & . & . & . & . & (\emptyset,2,\emptyset) \\
1 & . & . & 1 & . & (\emptyset,1,1) \\
. & 1 & . & . & . & (\emptyset,1^2,\emptyset) \\
. & . & . & 1 & . & (\emptyset,\emptyset,1^2)
\end{array}
\qquad
\begin{array}{cccccccc||l}
1 & . & . & . & . & . & . & . & (3,\emptyset,\emptyset) \\
. & 1 & . & . & . & . & . & . & (2,\emptyset,1) \\
. & . & 1 & . & . & . & . & . & (2,1,\emptyset) \\
1 & . & . & 1 & . & . & . & . & (2.1,\emptyset,\emptyset) \\
. & . & . & . & 1 & . & . & . & (\emptyset,\emptyset,3) \\
1 & . & . & . & 1 & 1 & . & . & (1,\emptyset,2) \\
. & . & . & . & . & . & 1 & . & (1^2,1,\emptyset) \\
. & . & 1 & . & . & . & . & 1 & (1,1,1) \\
1 & . & . & . & . & . & . & . & (\emptyset,3,\emptyset) \\
. & . & 1 & . & . & . & . & . & (1,2,\emptyset) \\
1 & . & . & 1 & . & 1 & . & . & (1^2,\emptyset,1) \\
1 & . & . & . & 1 & 1 & . & . & (\emptyset,1,2) \\
. & 1 & . & . & . & . & . & . & (\emptyset,2,1) \\
1 & . & . & 1 & . & . & . & . & (\emptyset,2.1,\emptyset) \\
. & . & . & . & 1 & 1 & . & . & (\emptyset,\emptyset,2.1) \\
. & . & . & . & . & . & 1 & . & (1,1^2,\emptyset) \\
. & . & . & 1 & . & . & . & . & (1^3,\emptyset,\emptyset) \\
. & . & . & . & . & . & . & 1 & (1,\emptyset,1^2) \\
1 & . & . & 1 & . & 1 & . & . & (\emptyset,1^2,1) \\
. & . & . & . & . & . & . & 1 & (\emptyset,1,1^2) \\
. & . & . & 1 & . & . & . & . & (\emptyset,1^3,\emptyset) \\
. & . & . & . & . & 1 & . & . & (\emptyset,\emptyset,1^3)
\end{array}
$$

- $e=3,\ \mathbf{s}=(0,1,2),\ n=1,2,3.$

$$
\begin{array}{ccc||l}
1 & . & . & (1,\emptyset,\emptyset) \\
. & 1 & . & (\emptyset,1,\emptyset) \\
. & . & 1 & (\emptyset,\emptyset,1)
\end{array}
\qquad
\begin{array}{cccccc||l}
1 & . & . & . & . & . & (2,\emptyset,\emptyset) \\
. & 1 & . & . & . & . & (\emptyset,2,\emptyset) \\
. & . & 1 & . & . & . & (\emptyset,\emptyset,2) \\
1 & . & . & 1 & . & . & (1,1,\emptyset) \\
. & . & 1 & . & 1 & . & (1,\emptyset,1) \\
. & 1 & . & . & . & 1 & (\emptyset,1,1) \\
. & . & . & . & 1 & . & (1^2,\emptyset,\emptyset) \\
. & . & . & 1 & . & . & (\emptyset,1^2,\emptyset) \\
. & . & . & . & . & 1 & (\emptyset,\emptyset,1^2)
\end{array}
$$

1	.	.	.	.	.	.	.	.	.	.	.	$(\emptyset, 3, \emptyset)$
.	1	.	.	.	.	.	.	.	.	.	.	$(\emptyset, \emptyset, 3)$
.	.	1	.	.	.	.	.	.	.	.	.	$(3, \emptyset, \emptyset)$
.	.	.	1	.	.	.	.	.	.	.	.	$(1, \emptyset, 2)$
.	.	.	.	1	.	.	.	.	.	.	.	$(2, 1, \emptyset)$
1	.	1	.	.	1	.	.	.	.	.	.	$(1, 2, \emptyset)$
.	1	1	.	.	.	1	.	.	.	.	.	$(2, \emptyset, 1)$
.	.	.	.	.	.	.	1	.	.	.	.	$(\emptyset, 2, 1)$
1	1	.	.	.	.	.	.	1	.	.	.	$(\emptyset, 1, 2)$
.	.	.	.	.	.	.	.	.	1	.	.	$(1^2, \emptyset, 1)$
.	.	.	.	.	.	.	.	.	.	1	.	$(1, 1^2, \emptyset)$
.	.	.	.	.	.	.	.	.	.	.	1	$(\emptyset, 1, 1^2)$
.	.	1	.	.	.	1	.	.	.	.	.	$(2.1, \emptyset, \emptyset)$
1	.	.	.	.	1	.	.	.	.	.	.	$(\emptyset, 2.1, \emptyset)$
.	1	.	.	.	.	.	.	1	.	.	.	$(\emptyset, \emptyset, 2.1)$
1	1	1	.	.	1	1	.	1	.	.	.	$(1, 1, 1)$
.	.	1	.	.	1	1	.	.	.	.	.	$(1^2, \emptyset, 1)$
.	1	.	.	.	.	1	.	1	.	.	.	$(\emptyset, 1^2, 1)$
1	.	.	.	.	1	.	.	1	.	.	.	$(\emptyset, 1^2, 1)$
.	.	.	.	.	1	.	.	.	.	.	.	$(\emptyset, 1^3, \emptyset)$
.	.	.	.	.	.	1	.	.	.	.	.	$(1^3, \emptyset, \emptyset)$
.	.	.	.	.	.	.	.	1	.	.	.	$(\emptyset, \emptyset, 1^3)$

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Notations

$\lvert X \rvert$	Cardinal of the set X.
$\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$	Set of integers, rational numbers, real numbers, complex numbers.
$\mathbb{C}^\times$	Non-zero complex numbers.
$\mathbb{Z}_{>a}$ or $\mathbb{Z}_{\geq a+1}$	Set of integers greater than a.
$[\![a, b]\!]$	Set of all integers between a and b.
$\lfloor x \rfloor$, $\{x\}$	Integral and fractional part of x.
$a \mod e$	Remainder of the euclidean division of a by e.
$A \otimes_R B$	Tensor product of A and B over R. When the underlying space R is understood, one drops the index.
$A \otimes B$	Outer tensor product.
$V^{\otimes n}$	n-th tensor power of V.
$\mathrm{char}(k)$	Characteristic of the field k.
$\mathrm{Frac}(A)$	Field of fractions of the ring A.
$A[X_1, \dots, X_n]$	Ring of polynomials over the ring A in the indeterminates $X_1, \dots, X_n$.
kG	Algebra of the group G over the field k.
$\mathrm{Irr}_k(G)$	Set of irreducible representations of the group G over the field k.
$\mathrm{Irr}(\mathcal{H})$	Set of simple modules of the algebra $\mathcal{H}$.

Index

Résumé :

Cette thèse est consacrée à l'étude des représentations modulaires des algèbres d'Ariki-Koike, et des liens avec la théorie des cristaux et des bases canoniques de Kashiwara via le théorème de catégorification d'Ariki.

Dans un premier temps, on étudie, grâce à des outils combinatoires, les matrices de décomposition de ces algèbres en généralisant les travaux de Geck et Jacon. On classifie entièrement les cas d'existence et de non-existence d'ensembles basiques, en construisant explicitement ces ensembles lorsqu'ils existent.

On explicite ensuite les isomorphismes de cristaux pour les représentations de Fock de l'algèbre affine quantique $\mathcal{U}_q(\widehat{\mathfrak{sl}_e})$. On construit alors un isomorphisme particulier, dit canonique, qui permet entre autres une caractérisation non-récursive de n'importe quelle composante connexe du cristal.

On souligne également les liens avec la combinatoire des mots sous-jacente à la structure cristalline des espaces de Fock, en décrivant notamment un analogue de la correspondance de Robinson-Schensted-Knuth pour le type A affine.

Mots clés :

Groupe symétrique, groupe de réflexions complexes, algèbre d'Ariki-Koike, représentations modulaires, matrice de décomposition, groupes quantiques, espace de Fock, cristaux, bases canoniques, correspondance RSK.

Abstract :

This thesis is devoted to the study of modular representations of Ariki-Koike algebras, and of the connections with Kashiwara's crystal and canonical bases theory via Ariki's categorification theorem.

First, we study, using combinatorial tools, the decomposition matrices associated to these algebras, generalising the works of Geck and Jacon. We fully classify the cases of existence and non-existence of canonical basic sets, and we explicitly construct these sets when they exist.

Next, we make explicit the crystal isomorphisms for Fock spaces representations of the quantum affine algebra $\mathcal{U}_q(\widehat{\mathfrak{sl}_e})$. We then construct of a particular isomorphism, so-called canonical, which gives, inter alia, a non-recursive description of any connected component of the crystal.

We also stress the links with the combinatorics of words underlying the crystal structure of Fock spaces, by describing notably an analogue of the Robinson-Schensted-Knuth correspondence for affine type A.

Keywords :

Symmetric group, complex reflection group, Ariki-Koike algebra, modular representations, decomposition matrix, quantum groups, Fock space, crystals, canonical bases, RSK correspondence.